Lecture 11 DETERMINANTS

Permutations and transpositions

. <u>permutation</u> : S_n = group of permutations of $\{1, \dots, n\}$

$\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ bijection

σ = permutation

. <u>transposition</u> : $k \neq \ell$, $\tau_{k\ell} \in S_n$

$$\tau_{k\ell} \downarrow \begin{pmatrix} 1 & \dots & k & \dots & \ell & \dots & n \\ 1 & \dots & \ell & \dots & k & \dots & n \end{pmatrix}$$

Exchange k and ℓ.

. <u>decomposition</u> any permutation σ can be decomposed as a product of transpositions :

$$\sigma = \prod_{i=1}^{n_\sigma} \tau_i \quad , \quad \tau_i \text{ transposition}$$

. <u>signature</u> of a permutation :

$$\varepsilon : S_n \longrightarrow \{1, -1\}$$
$$\sigma \longmapsto \begin{cases} 1 & \text{if } n_\sigma \text{ even} \\ -1 & \text{if } n_\sigma \text{ odd} \end{cases}$$

Then: ε is a homomorphism :

$$\varepsilon(\sigma \circ \sigma') = \varepsilon(\sigma)\,\varepsilon(\sigma')$$

Def $A \in M_n(F)$, $A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$.

We define the determinant of A :

$$\det A = \sum_{\sigma \in S_n}' \varepsilon(\sigma) \underbrace{a_{\sigma(1)1}\, a_{\sigma(2)2} \cdots a_{\sigma(n)n}}$$

$n!$ summands

one term for each row and column

Ex

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$\cdot \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\varepsilon = 1$

$\cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\varepsilon = -1$

Lemma If $A = (a_{ij})$ is an upper (lower) triangular matrix:

$$a_{ij} = 0 \text{ for } i > j \quad (\text{resp } i < j)$$

then: $\det A = 0$.

$$\det \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} = = \det \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} = 0$$

proof $\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$

For the summand not to be zero, I need:

$$\sigma(j) \leq j \quad \forall j$$

$$\Rightarrow \quad \sigma(i) = i \quad \forall i$$

but then $a_{ii} = 0$, $\det A = 0$ $\square$

Lemma $\det A = \det A^T$

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proof

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) \underbrace{\prod_{i=1}^{n} a_{\sigma(i)i}}_{\parallel}$$

σ bijection from $\{1, \dots, n\} \to \{1, \dots, n\}$

$$= \sum_{\sigma \in S_n} \underbrace{\varepsilon(\sigma)}_{\substack{\parallel \\ \varepsilon(\sigma^{-1})}} \underbrace{\prod_{i=1}^{n} a_{i\sigma^{-1}(i)}}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma^{-1}) \prod_{i=1}^{n} a_{i\sigma^{-1}(i)} = \overbrace{\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i\underline{\sigma(i)}}}^{\det A^T}$$

$\Rightarrow \quad \det A = \det A^T$ $\square$

Def A volume form d on F^n is a function:

$$\underbrace{F^n \times F^n \times \dots \times F^n}_{n} \longrightarrow F \text{ such that:}$$

(i) d multilinear: for any $1 \leq i \leq n$, $\forall$ $(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) \in \underbrace{F^n \times \dots \times F^n}_{n-1}$,

$$F^n \longrightarrow F$$
$$v \longmapsto d(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_n)$$

is linear $\left(\in (F^n)^* \right)$

(ii) d alternate: if $v_i = v_j$ for some $i \neq j$,

then: $\quad d(v_1, \dots, v_n) = 0$.

I want to show that there is in fact only one such volume form on $F^n \times \dots \times F^n$, and it is given by the determinant.

- $A = (a_{ij}) = \left(A^{(1)} | \dots | A^{(n)}\right)$

column vector 1 column vector n

Lemma $F^n \times \dots \times F^n \longrightarrow F$ is a volume form
$\left(A^{(1)}, \dots, A^{(n)}\right) \longmapsto \det A$

proof (i) Multilinear. Fix $\sigma \in S_n$, then:

$\left(\prod_{i=1}^{n} a_{\sigma(i)i}\right)$ is multilinear: there is only one term from each column appearing in this expression. Then the sum of multilinear maps is multilinear $\rightsquigarrow$ n.

(ii) Alternate $\quad k \neq \ell, \quad A^{(k)} = A^{(\ell)}$

Let τ = permutation which exchanges k and ℓ

$$\begin{pmatrix} 1 & \dots & k & \dots & \ell & \dots & n \\ \downarrow & & & & & & \downarrow \\ 1 & & \ell & & k & & n \end{pmatrix} \tau$$

Then : $\boxed{a_{ij} = a_{i\tau(j)}}$ $\quad \forall i, j \in \{1, \dots, n\}$

I can decompose :

$$S_n = \underbrace{A_n}_{\text{even \# transpositions}} \sqcup \tau A_n$$

bijection

disjoint union

$$\det A = \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod_{i=1}^{n} \underbrace{a_{i\tau\sigma(i)}}_{= a_{i\sigma(i)}}$$

$$= \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} - \sum_{\sigma \in A_n} \prod_{i=1}^{n} a_{i\sigma(i)} = 0 .$$

Lemma Let d be a volume form. Then swapping two entries changes the sign:

$$d(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n)$$

$$= - d(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n)$$

proof

$$d(v_1, \ldots, \underbrace{\boxed{v_i + v_j}}_{i\text{th position}}, \ldots, \underbrace{\boxed{v_i + v_j}}_{j\text{th position}}, \ldots, v_n) \underset{\text{alternate}}{=} 0$$

$$= d(v_1, \ldots, v_i, \ldots, v_i, \ldots, v_n) \quad = 0$$

$$+ d(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_j, \dots, v_n) = 0$$

$$\Rightarrow \quad d(v_1, \dots, v_i, \dots, v_j, \dots, v_n)$$

$$+ d(v_1, \dots, v_j, \dots, v_i, \dots, v_n) = 0 \qquad \square$$

Cor $\sigma \in S_n$, d volume form:

$$\boxed{d(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \varepsilon(\sigma)\, d(v_1, \dots, v_n)}$$

proof $\sigma = \prod_{i=1}^{n_\sigma} \tau_i$, $\tau_i =$ permutation. $\square$

Th Let d be a volume form on F^n.
Let $A = \left(A^{(1)} | \dots | A^{(n)}\right)$. Then:

$$d\left(A^{(1)} | \dots | A^{(n)}\right) = \det A \, \big(d(e_1, \dots, e_n) \big)$$

$$e_i = \begin{array}{l} 0 \\ \vdots \\ 1 \leftarrow i \\ 0 \\ \vdots \\ 0 \end{array} , \quad 1 \leq i \leq n.$$

$\leadsto$ Up to a constant, det is the <u>ONLY</u> volume form on F^n.

proof $d\left(A^{(1)}, \dots, A^{(n)}\right)$

$$= d\left(\sum_{i=1}^{n} a_{i1} e_i , A^{(2)}, \dots, A^{(n)} \right)$$

$$\underset{\text{linearity}}{=} \sum_{i=1}^{n} a_{i1} \, d\left(e_i, A^{(2)}, \dots, A^{(n)}\right)$$

$$= \sum_{i=1}^{n} a_{i1}\, d\left(e_i, \sum_{j=1}^{n} a_{j2} e_j, A^{(3)}, \dots, A^{(n)}\right)$$

$$= \sum_{i,j=1}^{n} a_{i1} a_{j2}\, d\left(e_i, e_j, A^{(3)}, \dots, A^{(n)}\right)$$

linearity

$$= \sum_{\substack{1 \le i_1 \le n \\ 1 \le i_2 \le n \\ \vdots \\ 1 \le i_n \le n}} \prod_{k=1}^{n} a_{i_k k}\, \underbrace{d\left(e_{i_1}, e_{i_2}, \dots, e_{i_n}\right)}$$

for this not to be zero I need all the i_k to be distinct:

$$\Leftrightarrow \exists \sigma \in S_n \,/\, i_k = \sigma(k)$$

$$= \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{\sigma(k)k}\, d\left(e_{\sigma(1)}, \dots, e_{\sigma(n)}\right)$$

alternate

$$= \sum_{\sigma \in S_n} \prod_{k=1}^{n} a_{\sigma(k)k}\, \varepsilon(\sigma)\, d\left(e_1, \dots, e_n\right)$$

$$= d(e_1, \dots, e_n) \left[\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{k=1}^{n} a_{\sigma(k) k} \right]$$

$$= d(e_1, \dots, e_n) \ \det A \qquad \square$$

$(*)$

$\boxed{\textbf{Cor} \quad \det \text{ is the unique volume form such that: } d(e_1, \dots, e_n) = 1.}$