## Linear Algebra: Example Sheet 3 of 4

1. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

- 2. By considering the rank or minimal polynomial of a suitable matrix, find the eigenvalues of the  $n \times n$  matrix A with each diagonal entry equal to  $\lambda$  and all other entries 1. Hence write down the determinant of A.
- 3. Let A be an  $n \times n$  matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.
- 4. (i) Let V be a vector space, let  $\pi_1, \pi_2, \dots, \pi_k$  be endomorphisms of V such that  $\mathrm{id}_V = \pi_1 + \dots + \pi_k$  and  $\pi_i \pi_j = 0$  for any  $i \neq j$ . Show that  $V = U_1 \oplus \dots \oplus U_k$ , where  $U_j = \mathrm{Im}(\pi_j)$ .
  - (ii) Let  $\alpha$  be an endomorphism of V satisfying the equation  $\alpha^3 = \alpha$ . By finding suitable endomorphisms of V depending on  $\alpha$ , use (i) to prove that  $V = V_0 \oplus V_1 \oplus V_{-1}$ , where  $V_{\lambda}$  is the  $\lambda$ -eigenspace of  $\alpha$ .
  - (iii) Apply (i) to prove the Generalised Eigenspace Decomposition Theorem stated in lectures. [Hint: It may help to re-read the proof of the Diagonalisability Theorem given in lectures. You may use the fact that if  $p_1(t), \ldots, p_k(t) \in \mathbb{C}[t]$  are non-zero polynomials with no common factor, Euclid's algorithm implies that there exist polynomials  $q_1(t), \ldots, q_k(t) \in \mathbb{C}[t]$  such that  $p_1(t)q_1(t) + \ldots + p_k(t)q_k(t) = 1$ .]
- 5. Let  $\alpha$  be an endomorphism of a complex vector space. Show that if  $\lambda$  is an eigenvalue for  $\alpha$  then  $\lambda^2$  is an eigenvalue for  $\alpha^2$ . Show further that every eigenvalue of  $\alpha^2$  arises in this way. Are the eigenspaces  $\operatorname{Ker}(\alpha \lambda \iota)$  and  $\operatorname{Ker}(\alpha^2 \lambda^2 \iota)$  necessarily the same?
- 6. (Cayley-Hamilton theorem for a general field) Let F be any field,  $A \in M_n(F)$ , and B = tI A. By considering the equation  $B \cdot adj(B) = det(B)I$ , or otherwise, prove that  $\chi_A(A) = 0$ , where  $\chi_A(t)$  is the characteristic polynomial of A.
- 7. Without appealing directly to the uniqueness of Jordan Normal Form show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix}
-2 & -2 & -1 \\
3 & 3 & 1 \\
3 & 2 & 2
\end{pmatrix}$$

similar to any of them? If so, which? Find a basis such that it is in Jordan Normal Form.

- 8. (a) Recall that the Jordan normal form of a  $3 \times 3$  complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for  $4 \times 4$  complex matrices.
  - (b) Let A be a  $5 \times 5$  complex matrix with  $A^4 = A^2 \neq A$ . What are the possible minimal polynomials of A? If A is not diagonalisable, what are the possible characteristic polynomials and JNFs of A?
- 9. Let V be a vector space of dimension n and  $\alpha$  an endomorphism of V with  $\alpha^n = 0$  but  $\alpha^{n-1} \neq 0$ . Without appealing to Jordan Normal Form, show that there is a vector y such that  $(y, \alpha(y), \alpha^2(y), \dots, \alpha^{n-1}(y))$  is a basis for V. What is the matrix representation of  $\alpha$  with respect to this basis? And the matrix representation of  $\alpha^k$ , for an arbitrary positive integer k?

Show that if  $\beta$  is an endomorphism of V which commutes with  $\alpha$ , then  $\beta = p(\alpha)$  for some polynomial p. [Hint: consider  $\beta(y)$ .] What is the form of the matrix for  $\beta$  with respect to the above basis?

- 10. (a) Let A be an invertible square matrix. Describe the eigenvalues and the characteristic and minimal polynomials of  $A^{-1}$  in terms of those of A.
  - (b) Prove that the inverse of a Jordan block  $J_m(\lambda)$  with  $\lambda \neq 0$  has Jordan Normal Form a Jordan block  $J_m(\lambda^{-1})$ . Use this to find the Jordan Normal Form of  $A^{-1}$ , for an invertible square matrix A.
  - (c) Prove that any square complex matrix is similar to its transpose.
- 11. Let C be an  $n \times n$  matrix over  $\mathbb{C}$ , and write C = A + iB, where A and B are real  $n \times n$  matrices. By considering  $\det(A + \lambda B)$  as a function of  $\lambda$ , show that if C is invertible then there exists a real number  $\lambda$  such that  $A + \lambda B$  is invertible. Deduce that if two  $n \times n$  real matrices P and Q are similar when regarded as matrices over  $\mathbb{C}$ , then they are similar as matrices over  $\mathbb{R}$ .
- 12. Let  $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ , with  $a_i \in \mathbb{C}$ , and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_n & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \dots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Show that the determinant of C is  $\det C = \prod_{j=0}^n f(\zeta^j)$ , where  $\zeta = \exp(2\pi i/(n+1))$ .

amk50@cam.ac.uk - 2 - November 2019