Linear Algebra: Example Sheet 2 of 4

1. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Use this method to find the inverse of
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

2. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

3. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

4. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

5. (a) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(b) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^\circ \geq B^\circ$. Show that $A = V$ if and only if $A^\circ = \{\mathbf{0}\}$.

6. For $A \in \mathrm{Mat}_{n,m}(\mathbb{F})$ and $B \in \mathrm{Mat}_{m,n}(\mathbb{F})$, let $\tau_A(B)$ denote $\mathrm{tr}AB$. Show that, for each fixed A, $\tau_A\colon \mathrm{Mat}_{m,n}(\mathbb{F}) \to \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathrm{Mat}_{n,m}(\mathbb{F}) \to \mathrm{Mat}_{m,n}(\mathbb{F})^*$.

7. (a) Let V be a non-zero finite dimensional real vector space. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

   (b) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α and β of V such that $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

8. Suppose that $\psi\colon U \times V \to \mathbb{F}$ is a bilinear form of rank r on finite dimensional vector spaces U and V over $\mathbb{F}$. Show that there exist bases $e_1, \ldots, e_m$ for U and $f_1, \ldots, f_n$ for V such that
$$\psi\left(\sum_{i=1}^m x_i e_i, \sum_{j=1}^n y_j f_j\right) = \sum_{k=1}^r x_k y_k$$
for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in \mathbb{F}$. What are the dimensions of the left and right kernels of ψ?

9. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

10. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A is singular? *[Hint: Consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]*

Show that the rank of the adjugate matrix is $\mathrm{r}(\mathrm{adj}\,A) = \begin{cases} n & \text{if } \mathrm{r}(A) = n \\ 1 & \text{if } \mathrm{r}(A) = n-1 \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

11. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

    In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
    (a) The map D defined by $D(p)(t) = p'(t)$.
    (b) The map S defined by $S(p)(t) = p(t^2)$.
    (c) The map E defined by $E(p)(t) = p(t-1)$.
    (d) The composite DS.
    (e) The composite SD.

    Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

12. Suppose that $\psi: V \times V \to \mathbb{F}$ is a bilinear form on a finite dimensional vector space V. Take U a subspace of V with $U = W^\perp$ some subspace W of V. Suppose that $\psi|_{U \times U}$ is non-singular. Show that ψ is also non-singular.

13. Let V be a vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that g is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{i=1}^n \ker f_i \subset \ker g$.

14. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
    (i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
    (ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

The final question is based on non-examinable material

15. Let Y and Z be subspaces of the finite dimensional vector spaces V and W respectively. Suppose that $\alpha: V \to W$ is a linear map such that $\alpha(Y) \subset Z$. Show that α induces linear maps $\alpha|_Y: Y \to Z$ via $\alpha|_Y(y) = \alpha(y)$ and $\overline{\alpha}: V/Y \to W/Z$ via $\overline{\alpha}(v + Y) = \alpha(v) + Z$.

    Consider a basis $(v_1, \ldots, v_n)$ for V containing a basis $(v_1, \ldots, v_k)$ for Y and a basis $(w_1, \ldots, w_m)$ for W containing a basis $(w_1, \ldots, w_l)$ for Z. Show that the matrix representing α with respect to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ is a block matrix of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Explain how to determine the matrices representing $\alpha|_Y$ with respect to the bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ and representing $\overline{\alpha}$ with respect to the bases $(v_{k+1} + Y, \ldots, v_n + Y)$ and $(w_{l+1} + Z, \ldots, w_m + Z)$ from this block matrix.