

## Linear Algebra: Example Sheet 1 of 4

1. Let  $\mathbb{R}^{\mathbb{R}}$  be the vector space of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of  $\mathbb{R}^{\mathbb{R}}$ ?
  - (a) The set  $C$  of continuous functions.
  - (b) The set  $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0, 1]\}$ .
  - (c) The set  $\{f \in C : f(t) \rightarrow 0 \text{ as } t \rightarrow \infty\}$ .
  - (d) The set  $\{f \in C : f(t) \rightarrow 1 \text{ as } t \rightarrow \infty\}$ .
  - (e) The set of solutions of the differential equation  $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4x(t) = 0$ .
  - (f) The set of solutions of  $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4x(t) = \sin t$ .
  - (g) The set of solutions of  $(\dot{x}(t))^2 - x(t) = 0$ .
  - (h) The set of solutions of  $(\ddot{x}(t))^4 + (x(t))^2 = 0$ .
2. Suppose that the vectors  $\mathbf{e}_1, \dots, \mathbf{e}_n$  form a basis for  $V$ . Which of the following are also bases?
  - (a)  $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \dots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$ ;
  - (b)  $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \dots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$ ;
  - (c)  $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \dots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$ .
3. Let  $T, U$  and  $W$  be subspaces of  $V$ .
  - (i) Show that  $T \cup U$  is a subspace of  $V$  only if either  $T \leq U$  or  $U \leq T$ .
  - (ii) Give explicit counter-examples to the following statements:
    - (a)  $T + (U \cap W) = (T + U) \cap (T + W)$ ;
    - (b)  $(T + U) \cap W = (T \cap W) + (U \cap W)$ .
  - (iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.
4. For each of the following pairs of vector spaces  $(V, W)$  over  $\mathbb{R}$ , either give an isomorphism  $V \rightarrow W$  or show that no such isomorphism can exist. [Here  $P$  denotes the space of polynomial functions  $\mathbb{R} \rightarrow \mathbb{R}$ , and  $C[a, b]$  denotes the space of continuous functions defined on the closed interval  $[a, b]$ .]
  - (a)  $V = \mathbb{R}^4$ ,  $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$ .
  - (b)  $V = \mathbb{R}^5$ ,  $W = \{p \in P : \deg p \leq 5\}$ .
  - (c)  $V = C[0, 1]$ ,  $W = C[-1, 1]$ .
  - (d)  $V = C[0, 1]$ ,  $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable}\}$ .
  - (e)  $V = \mathbb{R}^2$ ,  $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$ .
  - (f)  $V = \mathbb{R}^4$ ,  $W = C[0, 1]$ .
  - (g) (Harder:)  $V = P$ ,  $W = \mathbb{R}^{\mathbb{N}}$ .
5. (i) If  $\alpha$  and  $\beta$  are linear maps from  $U$  to  $V$  show that  $\alpha + \beta$  is linear. Give explicit counter-examples to the following statements:
  - (a)  $\text{Im}(\alpha + \beta) = \text{Im}(\alpha) + \text{Im}(\beta)$ ;
  - (b)  $\text{Ker}(\alpha + \beta) = \text{Ker}(\alpha) \cap \text{Ker}(\beta)$ .

Show that in general each of these equalities can be replaced by a valid inclusion of one side in the other.

(ii) Let  $\alpha$  be a linear map from  $V$  to  $V$ . Show that if  $\alpha^2 = \alpha$  then  $V = \text{Ker}(\alpha) \oplus \text{Im}(\alpha)$ . Does your proof still work if  $V$  is infinite dimensional? Is the result still true?

6. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0, 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0, x_2 = x_3 = x_4\}.$$

Find bases for  $U$  and  $W$  containing a basis for  $U \cap W$  as a subset. Give a basis for  $U + W$  and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

7. Let  $\alpha: U \rightarrow V$  be a linear map between two finite dimensional vector spaces and let  $W$  be a vector subspace of  $U$ . Show that the restriction of  $\alpha$  to  $W$  is a linear map  $\alpha|_W: W \rightarrow V$  which satisfies

$$r(\alpha) \geq r(\alpha|_W) \geq r(\alpha) - \dim(U) + \dim(W).$$

Give examples (with  $W \neq U$ ) to show that either of the two inequalities can be an equality.

8. (i) Let  $\alpha: V \rightarrow V$  be an endomorphism of a finite dimensional vector space  $V$ . Show that

$$V \supseteq \text{Im}(\alpha) \supseteq \text{Im}(\alpha^2) \supseteq \dots \quad \text{and} \quad \{0\} \subseteq \text{Ker}(\alpha) \subseteq \text{Ker}(\alpha^2) \subseteq \dots$$

If  $r_k = r(\alpha^k)$ , deduce that  $r_k \geq r_{k+1}$  and that  $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$ . Conclude that if, for some  $k \geq 0$ , we have  $r_k = r_{k+1}$ , then  $r_k = r_{k+\ell}$  for all  $\ell \geq 0$ .

(ii) Suppose that  $\dim(V) = 5$ ,  $\alpha^3 = 0$ , but  $\alpha^2 \neq 0$ . What possibilities are there for  $r(\alpha)$  and  $r(\alpha^2)$ ?

9. Let  $\alpha: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by  $\alpha: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ . Find the matrix

representing  $\alpha$  relative to the basis  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  for both the domain and the range.

Write down bases for the domain and range with respect to which the matrix of  $\alpha$  is the identity.

10. Let  $U_1, \dots, U_k$  be subspaces of a vector space  $V$  and let  $B_i$  be a basis for  $U_i$ . Show that the following statements are equivalent:

(i)  $U = \sum_i U_i$  is a direct sum, *i.e.* every element of  $U$  can be written uniquely as  $\sum_i u_i$  with  $u_i \in U_i$ .

(ii)  $U_j \cap \sum_{i \neq j} U_i = \{0\}$  for all  $j$ .

(iii) The  $B_i$  are pairwise disjoint and their union is a basis for  $\sum_i U_i$ .

Give an example where  $U_i \cap U_j = \{0\}$  for all  $i \neq j$ , yet  $U_1 + \dots + U_k$  is not a direct sum.

11. Let  $Y$  and  $Z$  be subspaces of the finite dimensional vector spaces  $V$  and  $W$ , respectively. Show that  $R = \{\alpha \in \mathcal{L}(V, W) : \alpha(Y) \subseteq Z\}$  is a subspace of the space  $\mathcal{L}(V, W)$  of all linear maps from  $V$  to  $W$ . What is the dimension of  $R$ ?

12. Recall that  $\mathbb{F}^n$  has standard basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . Let  $U$  be a subspace of  $\mathbb{F}^n$ . Show that there is a subset  $I$  of  $\{1, 2, \dots, n\}$  for which the subspace  $W = \langle \{\mathbf{e}_i : i \in I\} \rangle$  is a complementary subspace to  $U$  in  $\mathbb{F}^n$ .

13. Suppose  $X$  and  $Y$  are linearly independent subsets of a vector space  $V$ ; no member of  $X$  is expressible as a linear combination of members of  $Y$ , and no member of  $Y$  is expressible as a linear combination of members of  $X$ . Is the set  $X \cup Y$  necessarily linearly independent? Give a proof or counterexample.

14. Show that any two subspaces of the same dimension in a finite dimensional real vector space have a common complementary subspace.

15. Let  $T, U, V, W$  be vector spaces over  $\mathbb{F}$  and let  $\alpha: T \rightarrow U, \beta: V \rightarrow W$  be fixed linear maps. Show that the mapping  $\Phi: \mathcal{L}(U, V) \rightarrow \mathcal{L}(T, W)$  which sends  $\theta$  to  $\beta \circ \theta \circ \alpha$  is linear. If the spaces are finite-dimensional and  $\alpha$  and  $\beta$  have rank  $r$  and  $s$  respectively, find the rank of  $\Phi$ .