Linear Algebra: Example Sheet 3 of 4

1. Show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   similar to any of them? If so, which?

2. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

3. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
   (b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal and characteristic polynomials? If A is not diagonalisable, how many possible JNFs are there for A?

4. Let α be an endomorphism of the finite dimensional vector space V over $\mathbb{F}$, with characteristic polynomial $\chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $\det(\alpha) = (-1)^n c_0$ and $\operatorname{tr}(\alpha) = -c_{n-1}$.

5. Let α be an endomorphism of the finite-dimensional vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

6. Prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary invertible square matrix A, describe the Jordan normal form of A^{-1} in terms of that of A.

   Prove that any square complex matrix is similar to its transpose.

7. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Show that there is a vector y such that $\langle y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y) \rangle$ is a basis for V.

   Show that if β is an endomorphism of V which commutes with α, then $\beta = p(\alpha)$ for some polynomial p. *[Hint: consider $\beta(y)$.]* What is the form of the matrix for β with respect to the above basis?

8. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimal polynomial of A over the complex numbers has real coefficients.

9. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
   (a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
   (b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
   (c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
   (d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

10. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

11. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional complex vector space and let $\alpha^* : V^* \to V^*$ be its dual. Show that a complex number λ is an eigenvalue for α if and only if it is an eigenvalue for α^*. How are the algebraic and geometric multiplicities of λ for α and α^* related? How are the minimal and characteristic polynomials for α and α^* related?

12. (a) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(b) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$.

13. For $A \in \mathrm{Mat}_{n,m}(\mathbb{F})$ and $B \in \mathrm{Mat}_{m,n}(\mathbb{F})$, let $\tau_A(B)$ denote $\mathrm{tr}AB$. Show that, for each fixed A, $\tau_A\colon \mathrm{Mat}_{m,n}(\mathbb{F}) \to \mathbb{F}$ is linear. Show moreover that the mapping $A \mapsto \tau_A$ defines a linear isomorphism $\mathrm{Mat}_{n,m}(\mathbb{F}) \to \mathrm{Mat}_{m,n}(\mathbb{F})^*$.

14. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
(a) The map D defined by $D(p)(t) = p'(t)$.
(b) The map S defined by $S(p)(t) = p(t^2)$.
(c) The map E defined by $E(p)(t) = p(t-1)$.
(d) The composite DS.
(e) The composite SD.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

The remaining two questions are based on non-examinable material

15. Let V be a vector space of finite dimension over a field F. Let α be an endomorphism of V and let U be an α-invariant subspace of V ie a subspace such that $\alpha(U) \leq U$. Define $\overline{\alpha} \in \mathrm{End}(V/U)$ by $\overline{\alpha}(v + U) = \alpha(v) + U$. Check that $\overline{\alpha}$ is a well-defined endomorphism of V/U.

Consider a basis $\langle v_1, \ldots, v_n\rangle$ of V containing a basis $\langle v_1, \ldots, v_k\rangle$ of U. Show that the matrix of α with respect to $\langle v_1, \ldots, v_n\rangle$ is $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where A the matrix of the restriction $\alpha_U\colon U \to U$ of α to U with respect to $\langle v_1, \ldots, v_k\rangle$, and B the matrix of $\overline{\alpha}$ with respect to $\langle v_{k+1} + U, \ldots, v_n + U\rangle$. Deduce that $\chi_\alpha = \chi_{\alpha_U}\chi_{\overline{\alpha}}$.

16. (Another proof of the Cayley Hamilton Theorem.) Assume that the Cayley Hamilton Theorem holds for any endomorphism on any vector space over the field $\mathbb{F}$ of dimension less than n. Let V be a vector space of dimension n and let α be an endomorphism of V. If U is a proper α-invariant subspace of V, use the previous question and the induction hypothesis to show that $\chi_\alpha(\alpha) = 0$. If no such subspace exists, show that there exists a basis $\langle v, \alpha(v), \ldots \alpha^{n-1}(v)\rangle$ of V. Show that α has matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}$$

with respect to this basis, for suitable $a_i \in \mathbb{F}$. Show that $\chi_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and that $\chi_\alpha(\alpha)(v) = 0$. Deduce that $\chi_\alpha(\alpha) = 0$ as an element of $\mathrm{End}(V)$.