Michaelmas Term 2012

Linear Algebra: Example Sheet 2 of 4

The first thirteen questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part. Questions 7 and 8 are largely for revision.

1. (A proof that row rank equals column rank.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(F)$ and $C \in \mathrm{Mat}_{r,n}(F)$.
   Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

2. Write down the elementary matrices $I + \alpha E_{ij}$, and the permutation matrices s_{ij}, $i \neq j$, and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices and permutation matrices. Write the following matrices as products of elementary matrices and hence find their inverses.
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

3. Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. [*Note that the sum of all columns of A has all entries equal.*]

4. Let $\lambda_1, \ldots, \lambda_n \in F$. Show that the determinant
$$\det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} = \prod_{j<i} (\lambda_i - \lambda_j).$$
   This is called the 'Vandermonde determinant'. [*You can do this exercise very easily by observing the determinant is a polynomial in the variables λ_i which vanishes whenever two are equal; comparing degrees of RHS and LHS you can conclude that they agree upto multiplication by a constant. Then check the highest order term.*]

5. Let $v_1, \ldots, v_r$ be a basis for a subspace U in $\mathbb{R}^n$, and $w_1, \ldots, w_s$ be a basis for a subspace W in $\mathbb{R}^n$. Explain how to find a basis for $U + W$ and a basis for $U \cap W$.
   Check your method works for $U = \langle e_1, e_2 \rangle$, $W = \langle e_3, e_3 - e_1 \rangle$ in $\mathbb{R}^3$.
   [*It might be useful to recall you know how to find a basis for the kernel of a linear map.*]

6. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
   by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

7. (i) Let V be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.
   (ii) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

8. Compute the characteristic polynomials of the matrices
$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
   Which of the matrices are diagonalisable over $\mathbb{C}$? Which over $\mathbb{R}$?

9. Show that for a 'block upper triangular' matrix,

$$\det \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_r \end{pmatrix} = \prod_{i=1}^{r} \det A_i$$

where the A_i are square matrices.

10. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

11. Let A be a square complex matrix of finite order - that is, $A^m = I$ for some m. Show that A can be diagonalised. [*You can use a theorem.*]

12. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. [*The corresponding claim fails for real vector spaces.*] Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda I)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2 I)$ necessarily the same?

13. (Another proof of the Diagonalisability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonalisable.

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14. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

15. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the adjugate matrix is $\quad r(\mathrm{adj}\,A) = \begin{cases} n & \text{if } r(A) = n \\ 1 & \text{if } r(A) = n-1 \\ 0 & \text{if } r(A) \leq n-2. \end{cases}$

16. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_n & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

17. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.