Linear Algebra: Example Sheet 3 of 4

The first eleven questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they are intended to be attempted only after completion of the first part.

1. Show that none of the following matrices are similar:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

similar to any of them? If so, which?

2. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

3. (a) Recall that the Jordan normal form of a 3×3 complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices.
(b) Let A be a 5×5 complex matrix with $A^4 = A^2 \neq A$. What are the possible minimal and characteristic polynomials? How many possible JNFs are there for A? [*There are enough that you probably don't want to list all the possibilities.*]

4. Let α be an endomorphism of the finite dimensional vector space V over F, with characteristic polynomial $\chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $\det(\alpha) = (-1)^n c_0$ and $\mathrm{tr}(\alpha) = -c_{n-1}$.

5. Let α be an endomorphism of the finite-dimensional vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

6. Prove that any square complex matrix is similar to its transpose. [*You may want to check it first for a Jordan block matrix.*]
Prove that that the inverse of a Jordan block $J_m(\lambda)$ with $\lambda \neq 0$ has Jordan normal form a Jordan block $J_m(\lambda^{-1})$. For an arbitrary non-singular square matrix A, describe the Jordan normal form of A^{-1} in terms of that of A.

7. Let V be a complex vector space of dimension n and let α be an endomorphism of V with $\alpha^{n-1} \neq 0$ but $\alpha^n = 0$. Show that there is a vector $\mathbf{x} \in V$ for which $\mathbf{x}, \alpha(\mathbf{x}), \alpha^2(\mathbf{x}), \ldots, \alpha^{n-1}(\mathbf{x})$ is a basis for V. Give the matrix of α relative to this basis.
Let $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to this basis? What is the minimal polynomial for α? What are the eigenvalues and eigenvectors?
Show that if an endomorphism β of V commutes with α then $\beta = p(\alpha)$ for some polynomial $p(t)$.
[*It may help to consider* $\beta(\mathbf{x})$.]

8. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimal polynomial of A, over the complex numbers, has real coefficients.

9. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

10. Let P_n be the space of real polynomials of degree at most n. For $x \in \mathbb{R}$ define $\varepsilon_x \in P_n^*$ by $\varepsilon_x(p) = p(x)$. Show that $\varepsilon_0, \ldots, \varepsilon_n$ form a basis for P_n^*, and identify the basis of P_n to which it is dual.

11. (a) Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.
(b) Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* if and only if $\{\mathbf{v} \in V : f(\mathbf{v}) = 0 \text{ for all } f \in F\} = \{\mathbf{0}\}$.

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12. Let V be the vector space of all complex sequences (z_n) satisfying the difference equation $z_{n+2} = 3z_{n+1} - 2z_n$ for $n = 1, 2, \ldots$. Write down (without solving the difference equation) an obvious basis for V and hence determine its dimension. Show that the "shift" operator which sends a sequence $(z_1, z_2, z_3, \ldots)$ to $(z_2, z_3, z_4, \ldots)$ is an endomorphism on V. Find the matrix which represents this map with respect to your basis. Show that there is a basis for V with respect to which the map is represented by a diagonal matrix.

What happens if we replace the difference equation by $z_{n+2} = 2z_{n+1} - z_n$?

13. Let V be a vector space of finite dimension over a field F. Let α be an endomorphism of V and let U be an α-invariant subspace of V (so $\alpha(U) \leq U$). The quotient group $V/U = \{\mathbf{v} + U : \mathbf{v} \in V\}$ is a vector space under natural operations (called the quotient space). Write $\overline{V} = V/U$, $\overline{\mathbf{v}} = \mathbf{v} + U$, and define $\overline{\alpha} \in L(\overline{V})$ by $\overline{\alpha}(\overline{\mathbf{v}}) = \overline{\alpha(\mathbf{v})}$. Check that $\overline{\alpha}$ is a well-defined endomorphism of $\overline{V}$. Consider a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V containing a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of U. Show that the matrix of α with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is $A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$, with B the matrix of the restriction α_U of α to U with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and C the matrix of $\overline{\alpha}$ with respect to $\overline{\mathbf{v}_{k+1}}, \ldots, \overline{\mathbf{v}_n}$. Deduce that $\chi_\alpha = \chi_{\alpha_U} \chi_{\overline{\alpha}}$.

14. (Another proof of the Cayley Hamilton Theorem.) Assume that the Cayley Hamilton Theorem holds for any endomorphism on any vector space over the field F of dimension less than n. Let V be a vector space of dimension n and let α be an endomorphism of V. If U is a proper α-invariant subspace of V, use the previous question and the induction hypothesis to show that $\chi_\alpha(\alpha) = 0$. If no such subspace exists, show that there exists a basis $\mathbf{v}, \alpha(\mathbf{v}), \ldots \alpha^{n-1}(\mathbf{v})$ of V. Show that α has matrix

$$\begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}$$

with respect to this basis, for suitable $a_i \in F$. By expanding in the last column or otherwise, show that $\chi_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Show that $\chi_\alpha(\alpha)(\mathbf{v}) = \mathbf{0}$, and deduce that $\chi_\alpha(\alpha)$ is 0 on V.

15. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
(a) The map D defined by $D(p)(t) = p'(t)$.
(b) The map S defined by $S(p)(t) = p(t^2)$.
(c) The map E defined by $E(p)(t) = p(t-1)$.
(d) The composite DS.
(e) The composite SD.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

16. For A an $n \times m$ and B an $m \times n$ matrix over the field F, let $\tau_A(B)$ denote $\mathrm{tr}AB$.
Show that, for each fixed A, τ_A is a linear map $\mathrm{Mat}_{m,n}(F) \to F$.
Now consider the mapping $A \mapsto \tau_A$. Show that it is a linear isomorphism $\mathrm{Mat}_{n,m}(F) \to \mathrm{Mat}_{m,n}(F)^*$.

17. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional complex vector space and let $\alpha^* : V^* \to V^*$ be its dual. Show that a complex number λ is an eigenvalue for α if and only if it is an eigenvalue for α^*. How are the algebraic and geometric multiplicities of λ for α and α^* related? How are the minimal and characteristic polynomials for α and α^* related?