

**Linear Algebra: Example Sheet 2 of 4**

The first twelve questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part. Questions 7 and 8 are largely for revision.

1. (i) Let  $\alpha : V \rightarrow V$  be an endomorphism of a finite dimensional vector space  $V$ . Show that

$$V \supseteq \text{Im}(\alpha) \supseteq \text{Im}(\alpha^2) \supseteq \dots \quad \text{and} \quad \{0\} \subseteq \text{Ker}(\alpha) \subseteq \text{Ker}(\alpha^2) \subseteq \dots$$

If  $r_k = r(\alpha^k)$ , deduce that  $r_k \geq r_{k+1}$ . Show also that  $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$ . [Consider the restriction of  $\alpha$  to  $\text{Im}(\alpha^k)$ .] Deduce that if, for some  $k \geq 0$ , we have  $r_k = r_{k+1}$ , then  $r_k = r_{k+\ell}$  for all  $\ell \geq 0$ .

(ii) Suppose that  $\dim(V) = 5$ ,  $\alpha^3 = 0$ , but  $\alpha^2 \neq 0$ . What possibilities are there for  $r(\alpha)$  and  $r(\alpha^2)$ ?

2. (Another proof of the row rank column rank equality.) Let  $A$  be an  $m \times n$  matrix of (column) rank  $r$ . Show that  $r$  is the least integer for which  $A$  factorises as  $A = BC$  with  $B \in \text{Mat}_{m,r}(F)$  and  $C \in \text{Mat}_{r,n}(F)$ . Using the fact that  $(BC)^T = C^T B^T$ , deduce that the (column) rank of  $A^T$  equals  $r$ .
3. Write down the three types of elementary matrices and find their inverses. Show that an  $n \times n$  matrix  $A$  is invertible if and only if it can be written as a product of elementary matrices. Write the following matrices as products of elementary matrices and hence find their inverses.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

4. Let  $\lambda \in F$ . Evaluate the determinant of the  $n \times n$  matrix  $A$  with each diagonal entry equal to  $\lambda$  and all other entries 1. [Note that the sum of all columns of  $A$  has all entries equal.]
5. Let  $A$  and  $B$  be  $n \times n$  matrices over a field  $F$ . Show that the  $2n \times 2n$  matrix

$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$

by elementary row operations (which you should specify). By considering the determinants of  $C$  and  $D$ , obtain another proof that  $\det AB = \det A \det B$ .

6. (i) Let  $V$  be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms  $\alpha, \beta$  of  $V$  with  $\alpha\beta - \beta\alpha = \text{id}_V$ .
- (ii) Let  $V$  be the space of infinitely differentiable functions  $\mathbb{R} \rightarrow \mathbb{R}$ . Find endomorphisms  $\alpha, \beta$  of  $V$  which do satisfy  $\alpha\beta - \beta\alpha = \text{id}_V$ .
7. Compute the characteristic polynomials of the matrices

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Which of the matrices are diagonalisable over  $\mathbb{C}$ ? Which over  $\mathbb{R}$ ?

8. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second and third matrices commute; find a basis with respect to which they are both diagonal.

9. Let  $V$  be a vector space, let  $\pi_1, \pi_2, \dots, \pi_k$  be endomorphisms of  $V$  such that  $\text{id}_V = \pi_1 + \dots + \pi_k$  and  $\pi_i \pi_j = 0$  for any  $i \neq j$ . Show that  $V = U_1 \oplus \dots \oplus U_k$ , where  $U_j = \text{Im}(\pi_j)$ .  
Let  $\alpha$  be an endomorphism on the vector space  $V$ , satisfying the equation  $\alpha^3 = \alpha$ . Prove directly that  $V = V_0 \oplus V_1 \oplus V_{-1}$ , where  $V_\lambda$  is the  $\lambda$ -eigenspace of  $\alpha$ .

10. Let  $A$  be a square complex matrix of finite order - that is,  $A^m = I$  for some  $m$ . Show that  $A$  can be diagonalised. [You can use a theorem.]
11. Let  $\alpha$  be an endomorphism of a finite dimensional complex vector space. Show that if  $\lambda$  is an eigenvalue for  $\alpha$  then  $\lambda^2$  is an eigenvalue for  $\alpha^2$ . Show further that every eigenvalue of  $\alpha^2$  arises in this way. [The corresponding claim fails for real vector spaces.] Are the eigenspaces  $\text{Ker}(\alpha - \lambda I)$  and  $\text{Ker}(\alpha^2 - \lambda^2 I)$  necessarily the same?
12. (Another proof of the Diagonalisability Theorem.) Let  $V$  be a vector space of finite dimension. Show that if  $\alpha_1$  and  $\alpha_2$  are endomorphisms of  $V$ , then the nullity  $n(\alpha_1\alpha_2)$  satisfies  $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$ . Deduce that if  $\alpha$  is an endomorphism of  $V$  such that  $p(\alpha) = 0$  for some polynomial  $p(t)$  which is a product of distinct linear factors, then  $\alpha$  is diagonalisable.

13. Let  $C$  be an  $n \times n$  matrix over  $\mathbb{C}$ , and write  $C = A + iB$ , where  $A$  and  $B$  are real  $n \times n$  matrices. By considering  $\det(A + \lambda B)$  as a function of  $\lambda$ , show that if  $C$  is invertible then there exists a real number  $\lambda$  such that  $A + \lambda B$  is invertible. Deduce that if two  $n \times n$  real matrices  $P$  and  $Q$  are similar when regarded as matrices over  $\mathbb{C}$ , then they are similar as matrices over  $\mathbb{R}$ .

14. Let  $A$  be an  $n \times m$  matrix. Prove that if  $B$  is an  $m \times n$  matrix then

$$r(AB) \leq \min(r(A), r(B)).$$

At the start of each year the jovial and popular Dean of Muddling (pronounced Chumly) College organises  $m$  parties for the  $n$  students of the College. Each student is invited to exactly  $k$  parties, and every two students are invited to exactly one party in common. Naturally  $k \geq 2$ . Let  $P = (p_{ij})$  be the  $n \times m$  matrix defined by

$$p_{ij} = \begin{cases} 1 & \text{if student } i \text{ is invited to party } j \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the matrix  $PP^T$  and find its rank. Deduce that  $m \geq n$ .

After the Master's cat has been found dyed green, maroon and purple on successive nights, the other fellows insist that next year  $k = 1$ . Why does the proof above now fail, and what will, in fact, happen next year? (The answer required is mathematical rather than sociological in nature.)

15. Let  $A, B$  be  $n \times n$  matrices, where  $n \geq 2$ . Show that, if  $A$  and  $B$  are non-singular, then

$$(i) \text{adj}(AB) = \text{adj}(B)\text{adj}(A), \quad (ii) \det(\text{adj} A) = (\det A)^{n-1}, \quad (iii) \text{adj}(\text{adj} A) = (\det A)^{n-2}A.$$

What happens if  $A$  is singular?

Show that the rank of the adjugate matrix is  $r(\text{adj} A) = \begin{cases} n & \text{if } r(A) = n \\ 1 & \text{if } r(A) = n - 1 \\ 0 & \text{if } r(A) \leq n - 2. \end{cases}$

16. Let  $f(x) = a_0 + a_1x + \dots + a_nx^n$ , with  $a_i \in \mathbb{C}$ , and let  $C$  be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_n & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \dots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Show that the determinant of  $C$  is  $\det C = \prod_{j=0}^{n-1} f(\zeta^j)$ , where  $\zeta = \exp(2\pi i/(n+1))$ .

17. Let  $\alpha : V \rightarrow V$  be an endomorphism of a real finite dimensional vector space  $V$  with  $\text{tr}(\alpha) = 0$ .
- (i) Show that, if  $\alpha \neq 0$ , there is a vector  $\mathbf{v}$  with  $\mathbf{v}, \alpha(\mathbf{v})$  linearly independent. Deduce that there is a basis for  $V$  relative to which  $\alpha$  is represented by a matrix  $A$  with all of its diagonal entries equal to 0.
- (ii) Show that there are endomorphisms  $\beta, \gamma$  of  $V$  with  $\alpha = \beta\gamma - \gamma\beta$ .