Linear Algebra: Example Sheet 1 of 4

The first twelve questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part.

1. Let $\mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^{\mathbb{R}}$?
(a) The set C of continuous functions.
(b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0, 1]\}$.
(c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
(d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
(e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.
(f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.
(g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for V. Which of the following are also bases?
(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$;
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$;
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1$;
(d) $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$.

3. Let V be a vector space over a field F.
(i) Describe a procedure for picking vectors in V that produces **either** a finite basis for V **or** an infinite linearly independent subset of V.
(ii) Show that V is finite dimensional if and only if every linearly independent subset $S \subset V$ is finite.
(iii) Deduce that a subspace of a finite dimensional vector space is always finite dimensional.
[Although it is true that every vector space V has a basis, this is only proved in lectures for V finite dimensional. It would not be reasonable to quote the more general result in answering this question.]

4. Let T, U and W be subspaces of V.
(i) Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.
(ii) Give explicit counter-examples to the following statements:

   (a) $T + (U \cap W) = (T + U) \cap (T + W)$; (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.

   (iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.

5. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. *[Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.]*
(a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
(b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0, 1]$, $W = C[-1, 1]$.
(d) $V = C[0, 1]$, $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4$, $W = C[0, 1]$.
(g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

6. (i) If α and β are linear maps from U to V show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements:

   (a) $\mathrm{Im}(\alpha + \beta) = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta)$; (b) $\mathrm{Ker}(\alpha + \beta) = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta)$.

   Show that each of these equalities can be replaced by a valid inclusion of one side in the other.
(ii) Let α be a linear map from V to V. Show that if $\alpha^2 = \alpha$ then $V = \mathrm{Ker}(\alpha) \oplus \mathrm{Im}(\alpha)$. Does your proof still work if V is infinite dimensional? Is the result still true?

7. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ \ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ \ x_2 = x_3 = x_4\}.$$

Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}\,.$$

8. Recall that F^n has standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let U be a subspace of F^n. Show that there is a subset I of $\{1, 2, \ldots, n\}$ for which the subspace $W = \langle\{\mathbf{e}_i : i \in I\}\rangle$ is a complementary subspace to U in F^n.

9. Let $\alpha : U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W : W \to V$ which satisfies

$$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W)\ .$$

Give examples (with $W \neq U$) to show that either of the two inequalities can be an equality.

10. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.

Write down bases for the domain and range with respect to which the matrix of α is the identity.

11. Let $U_1, \ldots, U_k$ be subspaces of a vector space V and let B_i be a basis for U_i. Show that the following statements are equivalent:

    (i) $U = \sum_i U_i$ is a direct sum, *i.e.* every element of U can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.

    (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all j.

    (iii) The B_i are pairwise disjoint and their union is a basis for $\sum_i U_i$.

    Give an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

12. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\alpha \in L(V, W) : \alpha(Y) \leq Z\}$ is a subspace of the space $L(V, W)$ of all linear maps from V to W. What is the dimension of R?

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13. Let V be a vector space over F and let W be a subspace. Show that there is a natural way in which the quotient group V/W is a vector space over F. Show that if the dimension of V is finite, then so is the dimension of V/W, and

$$\dim V = \dim W + \dim V/W.$$

14. Suppose X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

15. Show that any two subspaces of the same dimension in a finite dimensional vector space have a common complementary subspace. *[You may wish to consider first the case where the subspaces have dimension one less than the space.]*

16. (Another version of the Steinitz Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite dimensional vector space have the same size.

17. Let $\mathbb{F}_p$ be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over $\mathbb{F}_p$. How many vectors are there in V? How many (ordered) bases? How many automorphisms does V have? How many k-dimensional subspaces are there in V?