Linear Algebra: Jordan Normal Form

One can regard the concrete proof of the existence of Jordan Normal Form (JNF) as consisting of three parts. First there is the decomposition into generalised eigenspaces. Then there is an analysis of (bases for) nilpotent endomorphisms. Finally we put things together to get the JNF.

In these notes V is a finite dimensional complex vector space.

1. Generalised Eigenspaces

The following decomposition is relatively straightforward to establish. (Essentially it depends on the Chinese Remainder Theorem.) Suppose that $\alpha : V \to V$ is an endomorphism of V; and suppose that its minimal polynomial is

$$m(t) = (t - \lambda_1)^{c_1}(t - \lambda_2)^{c_2} \dots (t - \lambda_k)^{c_k}$$

so we have k distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then V is the direct sum

$$V = V(\lambda_1) \oplus V(\lambda_2) \oplus \cdots \oplus V(\lambda_k)$$

of the *generalised eigenspaces*

$$V(\lambda_i) = \mathrm{Ker}((\alpha - \lambda_i \mathrm{id}_V)^{c_i}) \qquad \text{for } i = 1, 2, \dots, k.$$

Furthermore each generalised eigenspace $V(\lambda_i)$ is α-invariant, that is α maps $V(\lambda_i)$ to $V(\lambda_i)$, and the endomorphism

$$\alpha_i : V(\lambda_i) \to V(\lambda_i) \ ; \ \mathbf{v} \mapsto \alpha(\mathbf{v})$$

has minimal polynomial $(t - \lambda_i)^{c_i}$. In particular we see that each $\alpha_i - \lambda_i \mathrm{id}$ is nilpotent, so we are reduced to constructing a "good basis" for a nilpotent endomorphism.

2. Nilpotent Endomorphisms

Definition 2.1. Let α be an endomorphism of V. A subspace $U \leq V$ is α-cyclic if it is the span of $v, \alpha(v), \alpha^2(v), \dots$ for some vector $v \in V$.

Notice that an α-cyclic subspace U is necessarily α-invariant, i.e. $\alpha(U) \subset U$. The following is a special case of a theorem in the Lent Term IB course Groups, Rings and Modules. (It is closely related to the result that a finite abelian group is a product of cyclic groups.)

Theorem 2.2. *Let $\alpha \in \mathrm{End}(V)$. Then V is a direct sum of α-cyclic subspaces.*

If α is nilpotent (recall this means $\alpha^d = 0$ for some d) then it acts on each α-cyclic subspace as described in Question 7 of Example Sheet 3. So we can find a basis for V so that the matrix of α is in Jordan Normal Form, with each Jordan block corresponding to an α-cyclic subspace.

With a little more work, we can prove the existence of JNF for nilpotent endomorphisms without assuming Theorem **??**. The argument needs arranging with some care – essentially one has to look for the largest blocks first. We give the proof as an induction, with the following induction step.

Lemma 2.3. *Let $\alpha \in \mathrm{End}(V)$. Suppose that $\mathrm{Im}(\alpha)$ has a basis $\mathcal{B}_1$ of the form*

$$w_1, \alpha(w_1), \ldots, \alpha^{d_1-1}(w_1), \ldots, w_r, \alpha(w_r), \ldots \alpha^{d_r-1}(w_r)$$

where the d_i are positive integers and $\alpha^{d_i}(w_i) = 0$ for each i. Pick any $v_i \in V$ with $\alpha(v_i) = w_i$, and extend $\alpha^{d_1-1}(w_1), \ldots, \alpha^{d_r-1}(w_r)$ to a basis $\alpha^{d_1-1}(w_1), \ldots, \alpha^{d_r-1}(w_r)$, $u_1, \ldots, u_s$ for $\mathrm{Ker}(\alpha)$. Then $\mathcal{B} = \mathcal{B}_1 \cup \{u_1, \ldots, u_s\} \cup \{v_1, \ldots v_r\}$ is a basis for V.

Proof: We check that $\mathcal{B}$ is (i) linearly independent and (ii) has size $\dim V$.
(i) Suppose there is a dependence relation

$$\sum_{i=1}^{r} \sum_{j=0}^{d_i} \lambda_{ij} \alpha^j(v_i) + \sum_{k=1}^{s} \mu_k u_k = 0 \tag{1}$$

for some $\lambda_{ij}, \mu_k \in \mathbb{C}$. Then applying α gives a dependence relation involving only the vectors in $\mathcal{B}_1$. Since $\mathcal{B}_1$ is linearly independent, we deduce $\lambda_{ij} = 0$ for all $j < d_i$. But then (**??**) only involves the vectors in our basis for $\mathrm{Ker}(\alpha)$. So it must be the trivial dependence relation.
(ii) By rank-nullity $\dim V = n(\alpha) + r(\alpha) = r + s + |\mathcal{B}_1| = |\mathcal{B}|$. □

Theorem 2.4. *Let $\alpha \in \mathrm{End}(V)$ be nilpotent. Then V has a basis $e_1, \ldots, e_n$ with the property that for each i, $\alpha(e_i)$ is either e_{i+1} or 0.*

Proof: The proof is by induction on $n = \dim(V)$. If $n = 0$ then $V = \{0\}$ with basis the empty set, and the statement is clear. So now suppose $n \geq 1$. Let $W = \mathrm{Im}(\alpha)$. Then α acts on W, and so restricts to an endomorphism of W (this restriction is still nilpotent). Since α is nilpotent, it cannot be an isomorphism and so $\dim W < \dim V$. We can now use the induction hypothesis to pick a basis for W, and extend it to a basis for V as described in Lemma **??**. Up to re-ordering, this basis has the property we want. □

Note: Our convention when defining JNF was to put the 1's above (rather than below) the diagonal. So we should now reverse the order of the basis in Theorem **??**.

3. Jordan Normal Form

Having analysed the nilpotent case, we return to the case of a general α as in Section 1. We apply what we have learnt about nilpotent endomorphisms in Section 2 to the nilpotent endomorphisms $\alpha_i - \lambda_i \mathrm{id}$. The matrix we get for α is of the form

$$\begin{pmatrix} B_1 & 0 & 0 & \ldots & 0 \\ 0 & B_2 & 0 & \ldots & 0 \\ 0 & 0 & B_3 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B_k \end{pmatrix}.$$

where the blocks B corresponding to the generalised eigenspaces $V(\lambda)$, are themselves of form

$$\begin{pmatrix} C_1 & 0 & 0 & \ldots & 0 \\ 0 & C_2 & 0 & \ldots & 0 \\ 0 & 0 & C_3 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & C_m \end{pmatrix}.$$

when we take the bases for each $V(\lambda)$ found in Section 2, so that the C_i are of the form $J_d(\lambda) = \lambda I + J_d(0)$. This gives the *Jordan Normal Form* for α.

4. A worked example

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

One can easily check that the characteristic polynomial is $(t-2)^6$, so there is just one eigenvalue 2. So we consider

$$A - 2I = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix};$$

then

$$(A - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

and

$$(A - 2I)^3 = 0.$$

Just by looking at the nullities we can see that there will be cyclic subspaces of dimensions 3, 2 and 1 in the JNF. Let $\alpha : \mathbb{C}^6 \to \mathbb{C}^6$ be the linear map $\mathbf{x} \mapsto A\mathbf{x}$.

First we find a generator for a cyclic subspace of dimension 3. We either see that $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \mathrm{Im}(\alpha - 2)^2$ and take a preimage $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ say; or we pick perhaps less obviously $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ to generate a complement to $\mathrm{Ker}(\alpha-2)^2$. So for a cyclic subspace of dimension 3 we get a basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with the last an eigenvector.

Next we find a generator for a cyclic subspace of dimension 2. Either we look in $\mathrm{Im}(\alpha - 2)$ where we already have two linearly independent vectors one an eigenvector; we seek a further vector which is also an eigenvector and get most obviously $\begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$, with preimage $\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$. Alternatively we find the same vector generating, together with the vector $\begin{pmatrix}1\\1\\-1\\1\\0\\1\end{pmatrix}$ which we already have, a complement to $\mathrm{Ker}(\alpha-2)$ in $\mathrm{Ker}(\alpha-2)^2$. So for a cyclic subspace of dimension 2 we get a basis

$$\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$$

with the last an eigenvector.

Finally we seek a generator for a cyclic subspace of dimension 1. So we either look in $\mathbb{C}^6$ where we already have five independent vectors and find a sixth which is an eigenvector; or else we look for a basis of $\mathrm{Ker}(\alpha-2)$ including the two eigenvectors $\begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix}$ we already have. Much the same either way, $\begin{pmatrix}0\\0\\0\\1\\0\\1\end{pmatrix}$ seems indicated. It generates a cyclic subspace of dimension 1.

In summary we have a basis (with $(\alpha-2)$-action indicated)

$$\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\\-1\\1\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\-1\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\0\\0\\1\\0\\1\end{pmatrix}$$

with respect to which α has matrix

$$\begin{pmatrix}2&0&0&0&0&0\\1&2&0&0&0&0\\0&1&2&0&0&0\\0&0&0&2&0&0\\0&0&0&1&2&0\\0&0&0&0&0&2\end{pmatrix}.$$

(Again, reversing the order of the basis puts the 1's above the diagonal.)