

Linear Algebra: Example Sheet 4 of 4

The first ten questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they are intended to be attempted only after completion of the first part.

1. The square matrices A and B over the field F are congruent if $B = P^TAP$ for some invertible matrix P over F . Which of the following symmetric matrices are congruent to the identity matrix over \mathbb{R} , and which over \mathbb{C} ? (Which, if any, over \mathbb{Q} ?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over \mathbb{R} .

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

If A is the matrix of the first of these (say), find a non-singular matrix P such that P^TAP is diagonal with entries ± 1 .

3. (i) Show that the function $\psi(A, B) = \text{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\text{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\text{tr}(AB^T)| \leq \text{tr}(AA^T)^{1/2} \text{tr}(BB^T)^{1/2}$.
 (ii) Show that the map $A \mapsto \text{tr}(A^2)$ is a quadratic form on $\text{Mat}_n(\mathbb{R})$. Find its rank and signature.
4. Let $\psi : V \times V \rightarrow \mathbb{C}$ be a Hermitian form on a complex vector space V .
 (i) Show that if $n > 2$ then $\psi(u, v) = \frac{1}{n} \sum_{k=1}^n \zeta^k \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.
 (ii) Find the rank and signature of ψ in the case $V = \mathbb{C}^3$ and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

5. Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite. Compute the basis of \mathbb{R}^3 obtained by applying the Gram-Schmidt process to the standard basis.
6. Let $W \leq V$ with V an inner product space. An endomorphism π of V is called an idempotent if $\pi^2 = \pi$. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.
7. Let S be an $n \times n$ real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.
8. An endomorphism α of a finite dimensional inner product space V is *positive semi-definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in V$. Prove that a positive semi-definite endomorphism has a unique positive semi-definite square root. [To prove uniqueness it may help first to show that any square root of α acts on the eigenspaces of α .]
9. Let V be a finite dimensional complex inner product space, let α be an endomorphism on V . Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector \mathbf{v} , and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^* . Deduce that there is an orthonormal basis of eigenvectors of α .
10. Find a linear transformation which reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \quad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \quad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2, \quad 2xy$ simultaneously to diagonal forms?

11. Let $f_1, \dots, f_t, f_{t+1}, \dots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V . Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \dots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \dots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V . Suppose Q has rank $p + q$ and signature $p - q$. Show that $p \leq t$ and $q \leq u$.
12. Suppose that Q is a non-degenerate quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \leq V$ with $\dim U = m$. What is the signature of Q ? Establish the following.
- There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \dots + x_{2m-1}x_{2m}$.
 - We can write $V = U \oplus W$ with Q also vanishing on W .
13. Let S be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Find the eigenvalues of the endomorphism of $\text{Mat}_n(\mathbb{R})$ given by $X \mapsto SX^T S$.
14. Suppose that α is an orthogonal endomorphism on the finite-dimensional real inner product space V . Prove that V can be decomposed into a direct sum of mutually orthogonal α -invariant subspaces of dimension 1 or 2. Determine the possible matrices of α with respect to orthonormal bases in the cases where V has dimension 1 or dimension 2.
15. Let V be a finite dimensional complex inner product space and let α be an invertible endomorphism on V . By considering $\alpha^* \alpha$, show that α can be factored as $\beta \gamma$ with β unitary and γ positive definite. [A self-adjoint endomorphism γ is *positive definite* if $\langle \gamma \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \neq 0$.]
16. Show that if A is an $m \times n$ real matrix of rank n then $A^T A$ is invertible. Is there a corresponding result for complex matrices?
17. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix with $|a_{ij}| \leq k$, then

$$|\det A| \leq k^n n^{n/2}.$$

18. Let P_n be the $(n + 1)$ -dimensional space of real polynomials of degree $\leq n$. Define

$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt.$$

Show that $\langle \cdot, \cdot \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \rightarrow P_n$ defined by

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint. What are the eigenvalues of α ?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.

- For $i \neq j$, $\langle s_i, s_j \rangle = 0$.
- s_0, \dots, s_n forms a basis for P_n .
- For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k .
- s_k is an eigenvector of α . (Give its eigenvalue.)

What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence $1, x, x^2, x^3$ and so on? (Calculate the first few terms?)

19. Let a_1, a_2, \dots, a_n be real numbers such that $a_1 + \dots + a_n = 0$ and $a_1^2 + \dots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n + a_na_1$?
20. Let A be a $2n \times 2n$ alternating matrix over a field F . Show that the determinant of A is a square. In fact $\det(A) = \text{pf}(A)^2$ where $\text{pf}(A)$ is a homogeneous polynomial of degree n in the entries of A (called the *Pfaffian* of A). Assuming this fact, show that every matrix in the *symplectic group*

$$\text{Sp}_{2n}(F) = \{P \in \text{GL}_{2n}(F) \mid P^T J P = J\}, \quad \text{where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

has determinant $+1$.