Linear Algebra: Example Sheet 2 of 4

The first twelve questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part. Questions 7 and 8 are largely for revision.

1. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Show that
$$V \geq \mathrm{Im}(\alpha) \geq \mathrm{Im}(\alpha^2) \geq \dots \quad \text{and} \quad \{0\} \leq \mathrm{Ker}(\alpha) \leq \mathrm{Ker}(\alpha^2) \leq \dots .$$
If $r_k = r(\alpha^k)$, deduce that $r_k \geq r_{k+1}$. Show also that $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$. [*Consider the restriction of* α *to* $\mathrm{Im}(\alpha^k)$.] Deduce that if, for some $k \geq 0$, we have $r_k = r_{k+1}$, then $r_k = r_{k+\ell}$ for all $\ell \geq 0$.
(ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $r(\alpha)$ and $r(\alpha^2)$?

2. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(F)$ and $C \in \mathrm{Mat}_{r,n}(F)$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

3. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Write the following matrices as products of elementary matrices and hence find their inverses.
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 0 \end{pmatrix}.$$

4. Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. [*Note that the sum of all columns of A has all entries equal.*]

5. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

6. (i) Let V be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.
(ii) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

7. Compute the characteristic polynomials of the matrices
$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Which of the matrices are diagonalisable over $\mathbb{C}$? Which over $\mathbb{R}$?

8. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$
The second and third matrices commute; find a basis with respect to which they are both diagonal.

9. Let V be a vector space, let $\pi_1, \pi_2, \dots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \dots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \dots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.
Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

10. Let A be a square complex matrix of finite order - that is, $A^m = I$ for some m. Show that A can be diagonalised. [*You can use a theorem.*]

11. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. [*The corresponding claim fails for real vector spaces.*] Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda I)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2 I)$ necessarily the same?

12. (Another proof of the Diagonalisability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonalisable.

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13. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

14. Let A be an $n \times m$ matrix. Prove that if B is an $m \times n$ matrix then

$$r(AB) \leq \min(r(A), r(B)).$$

At the start of each year the jovial and popular Dean of Muddling (pronounced Chumly) College organises m parties for the n students of the College. Each student is invited to exactly k parties, and every two students are invited to exactly one party in common. Naturally $k \geq 2$. Let $P = (p_{ij})$ be the $n \times m$ matrix defined by

$$p_{ij} = \begin{cases} 1 & \text{if student } i \text{ is invited to party } j \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the matrix PP^T and find its rank. Deduce that $m \geq n$.
After the Master's cat has been found dyed green, maroon and purple on successive nights, the other fellows insist that next year $k = 1$. Why does the proof above now fail, and what will, in fact, happen next year? (The answer required is mathematical rather than sociological in nature.)

15. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the adjugate matrix is $\quad r(\mathrm{adj}\,A) = \begin{cases} n & \text{if } r(A) = n \\ 1 & \text{if } r(A) = n-1 \\ 0 & \text{if } r(A) \leq n-2. \end{cases}$

16. Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

17. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.