Linear Algebra: Preliminaries

This is NOT one of the official examples sheets of the course.

This sheet contains a few questions to revise some of the linear algebra covered in the IA Vectors and Matrices course last year. It is not one of the official example sheets - there will be four of these.

You need to recognise when you are and when you are not confronted by a linear situation. So I give examples and non-examples of spaces and of linear maps, with a little work on bases and dimension thrown in. At the start of the course it is important to develop clear intuitions about the answers. Work them out, in groups if you prefer, over the first weekend. There is little to be gained from writing out all the solutions, but you need to be able to do this kind of thing if required. So write out some answers for practice.

A page of notes for revision, compiled by a previous lecturer, Professor Hyland, appears on the website for this course, under the heading Recapitulation.

1. Let U be the subset of $\mathbb{R}^3$ consisting of all vectors $\mathbf{x}$ satisfying the various conditions below. In which of these cases is U a vector space over $\mathbb{R}$?
(a) $x_1 > 0$.
(b) either $x_1 = 0$ or $x_2 = 0$.
(c) $x_1 + x_2 = 0$.
(d) $x_1 + x_2 = 1$.
(e) $x_1 + x_2 + x_3 = 0$ and $x_1 - x_3 = 0$.

2. Determine which of the following sets of sequences of real numbers (x_n) form vector spaces over $\mathbb{R}$.
(a) x_n is bounded.
(b) x_n is convergent.
(c) $x_n \to 1$ as $n \to \infty$.
(d) $x_n \to 0$ as $n \to \infty$.
(e) $x_{n+2} = x_{n+1} + x_n$.
(f) There exists N such that $x_n = 0$ for $n > N$.
(g) $\sum |x_n|$ is convergent.
(h) $\sum x_n^2$ is convergent.

3. Let P be the vector space of all real polynomials and U the subset consisting of all polynomials f satisfying the various conditions below. In which of these cases is U a subspace of P?
(a) f has degree 3.
(b) f has degree ≤ 3.
(c) f has even degree.
(d) $2f(0) = f(1)$.
(e) $f(t) = f(1-t)$.

4. For each of the vector spaces found in questions 1, 2 and 3 determine whether it is finite dimensional or infinite dimensional. When it is finite dimensional what is the dimension? Can you give a basis?

5. Show that the four vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$ form a linearly dependent set, but that any proper subset of them is linearly independent.

6. Which of the following are bases for $\mathbb{R}^3$?

$$\text{(a)} \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}; \qquad \text{(b)} \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

7. Which of the following are bases for $\mathbb{R}^4$?

$$\text{(a)} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}; \qquad \text{(b)} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}.$$

8. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.

$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad ; \quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix} \quad ; \quad \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}.$$

9. Let P denote the space of all polynomial functions $\mathbb{R} \to \mathbb{R}$. Which of the following define linear maps $P \to P$?
(a) $D(p)(t) = p'(t)$.
(b) $S(p)(t) = p(t^2 + 1)$.
(c) $T(p)(t) = p(t)^2 + 1$.
(d) $E(p)(t) = p(e^t)$.
(e) $J(p)(t) = \int_0^t p(s)\,ds$.
(f) $K(p)(t) = 1 + \int_0^t p(s)\,ds$.
(g) $L(p)(t) = p(0) + \int_0^t p(s)\,ds$.
(h) $M(p)(t) = p(t^2) - tp(t)$.
(i) $R(p)$ is the remainder when the polynomial p is divided by the fixed polynomial $t^2 + 1$.
(j) $Q(p)$ is the quotient when the polynomial p is divided by the fixed polynomial $t^2 + 1$.

10. For each part of the previous question where the answer is 'yes', find the rank and nullity of the linear map $P_5 \to P$ (where P_5 denotes the space of polynomials of degree at most 5) obtained by restricting the given linear map to the vector subspace P_5 of P.

When you come to write out solutions it may be helpful to keep the following in mind.

- *To show that V is not a vector space identify an instance of a failure of an axiom (i.e. of some aspect of linearity). To show that U is not a subspace of a space V identify a failure of closure.*
- *To show V is a vector space one generally identifies it as a subspace of some standard space F^X of all functions $X \to F$: so V needs to be non-empty and closed under the operations.*
- *To test for linearity of α it is generally best to check $\alpha(\lambda.\mathbf{u} + \mu.\mathbf{v}) = \lambda.\alpha(\mathbf{u}) + \mu.\alpha(\mathbf{v})$. Just occasionally it is easier to check $\alpha(\mathbf{u} + \mathbf{v}) = \alpha(\mathbf{u}) + \alpha(\mathbf{v})$ and $\alpha(\lambda.\mathbf{v}) = \lambda.\alpha(\mathbf{v})$ separately.*
- *To show linear dependence it suffices to write one of the vectors in terms of the others.*
- *The definition implies that to show linear independence you should take a linear combination $\sum_i \lambda_i \mathbf{x}_i = \mathbf{0}$ and show directly that all $\lambda_i = 0$. But it can be easier to argue as follows. Suppose that you have k vectors, and you can show they span a vector space which you independently know is of dimension k. Then they form a basis of that space and so in particular are linearly independent.*
- *The rank $r(\alpha)$ of a linear map $\alpha : V \to W$ is the dimension of the image, and the nullity $n(\alpha)$ is the dimension of the kernel. The rank-nullity theorem states that (for V finite dimensional) the rank and nullity are related by $r(\alpha) + n(\alpha) = \dim V$.*