Linear Algebra: Example Sheet 1 of 4

The first twelve questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part.

1. Let $\mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^{\mathbb{R}}$?
(a) The set C of continuous functions.
(b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.
(c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
(d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
(e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.
(f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.
(g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for V. Which of the following are also bases?
(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$;
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$;
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1$;
(d) $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$.

3. Show that a vector space V is finite dimensional if and only if every linearly independent subset $S \subset V$ is finite. Deduce that a subspace of a finite dimensional vector space is always finite dimensional. [Although it is true that every vector space V has a basis, we only proved this in lectures for V finite dimensional, so it would not be reasonable to quote the more general result in answering this question.]

4. Let T, U and W be subspaces of V.
(i) Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.
(ii) Give explicit counter-examples to the following statements:
$$\text{(a)}\quad T + (U \cap W) = (T + U) \cap (T + W); \qquad \text{(b)}\quad (T + U) \cap W = (T \cap W) + (U \cap W).$$
(iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.

5. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a,b]$ denotes the space of continuous functions defined on the closed interval $[a,b]$.)
(a) $V = \mathbb{R}^4,\ W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
(b) $V = \mathbb{R}^5,\ W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0,1],\ W = C[-1,1]$.
(d) $V = C[0,1],\ W = \{f \in C[0,1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2,\ W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4,\ W = C[0,1]$.
(g) (Harder:) $V = P,\ W = \mathbb{R}^{\mathbb{N}}$.

6. If α and β are linear maps from U to V show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements:
$$\text{(a)}\quad \mathrm{Im}(\alpha + \beta) = \mathrm{Im}(\alpha) + \mathrm{Im}(\beta); \qquad \text{(b)}\quad \mathrm{Ker}(\alpha + \beta) = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(\beta).$$
Show that each of these equalities can be replaced by a valid inclusion of one side in the other.

7. Let
$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}.$$
Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that
$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

8. (i) Let U be a subspace of F^n. Show that there is a subset I of $\{1, 2, \ldots, n\}$ for which the subspace $W = \langle \{\mathbf{e}_i : i \in I\} \rangle$ is a complementary subspace to U in F^n.
(ii) Show that any two subspaces of the same dimension in a finite-dimensional vector space have a common complementary subspace. [You may wish to consider first the case where the subspaces have dimension 1 less than the space.]

9. Let $\alpha : U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W : W \to V$ which satisfies
$$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W) \ .$$
Give examples (with $W \neq U$) to show that either of the two inequalities can be an equality.

10. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

11. Let $U_1, \ldots, U_k$ be subspaces of a vector space V and let B_i be a basis for U_i. Show that the following statements are equivalent:
   (i) $U = \sum_i U_i$ is a direct sum, *i.e.* every element of U can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.
   (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all j.
   (iii) The B_i are pairwise disjoint and their union is a basis for $\sum_i U_i$.

   Given an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

12. Let U and V be finite dimensional vector spaces over F, with U not the zero space. Let V_1 and V_2 be subspaces of V. Show that $W_i = \{\alpha \in L(U, V) : \alpha(U) \leq V_i\}$ is a subspace of $L(U, V)$. Prove that $V = V_1 \oplus V_2$ if and only if $L(U, V) = W_1 \oplus W_2$.

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13. X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

14. Let U be a proper subspace of the finite-dimensional vector space V. Show there is a basis for V containing no element of U.

15. Let $\mathrm{Mat}_{m,n}(F)$ be the space of m by n matrices over a field F. We say $A, B \in \mathrm{Mat}_{m,n}(F)$ are *equivalent* if there exist invertible matrices P and Q such that $B = Q^{-1}AP$. Check this is an equivalence relation. Show that matrices are equivalent if and only if they have the same rank. (The "only if" part will be proved in lectures.)

16. (Another version of the Steinitz Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite-dimensional vector space have the same size.

17. Let $\mathbb{F}_p$ be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over $\mathbb{F}_p$. How many vectors are there in V? How many (ordered) bases? How many automorphisms does V have? How many k-dimensional subspaces are there in V?