Michaelmas Term 2008 T.A. Fisher

Linear Algebra: Non-degenerate Bilinear Forms

These notes cover some material related to the linear algebra course, marginally beyond that specified in the schedules. This includes the classification of skew-symmetric bilinear forms (recall that symmetric bilinear forms were covered in lectures). The last section of the course is on inner products, *i.e.* positive definite symmetric bilinear forms (case $F = \mathbb{R}$), respectively positive definite Hermitian forms (case $F = \mathbb{C}$). In these notes we generalise some of the results of that section to non-degenerate forms.

1. Non-degeneracy

Let V and W be finite dimensional vector spaces over a field F. Recall that $V^* = L(V, F)$ is the dual space of V. If $\psi : V \times W \to F$ is a bilinear form then there are linear maps

$$\begin{aligned} \psi_L : V \to W^* \, ; &\quad v \mapsto (w \mapsto \psi(v, w)) \\ \psi_R : W \to V^* \, ; &\quad w \mapsto (v \mapsto \psi(v, w)). \end{aligned}$$

Linearity of ψ in the second argument shows that $\psi_L(v) : W \to F$ is linear, and hence an element of W^*, whereas linearity of ψ in the first argument shows that ψ_L itself is linear. (The same comments apply to ψ_R with obvious modifications.)

Theorem 1.1. *Any two of the following statements implies the third.*

(i) $\mathrm{Ker}(\psi_L) = \{0\}$, *i.e.* $\psi(v, w) = 0$ *for all* $w \in W$ *implies* $v = 0$.
(ii) $\mathrm{Ker}(\psi_R) = \{0\}$, *i.e.* $\psi(v, w) = 0$ *for all* $v \in V$ *implies* $w = 0$.
(iii) $\dim V = \dim W$.

Proof: Statement (i) shows that $\dim V \leq \dim W^* = \dim W$, and likewise (ii) shows that $\dim W \leq \dim V^* = \dim V$. So (i) and (ii) imply $\dim V = \dim W$.

Now suppose that (i) and (iii) hold. Then $\psi_L : V \to W^*$ is an isomorphism. Pick a basis $v_1, \ldots, v_n$ for V. Then $\psi_L(v_1), \ldots, \psi_L(v_n)$ is a basis for W^*. Let $w_1, \ldots, w_n$ be the dual basis for W. Then $\psi(v_i, w_j) = \psi_L(v_i)(w_j) = \delta_{ij}$. If $w \in \mathrm{Ker}(\psi_R)$, say $w = \sum \lambda_j w_j$ for some $\lambda_j \in F$, then $\lambda_i = \psi(v_i, w) = \psi_R(w)(v_i) = 0$ for all i. Hence $w = 0$ and this proves (ii). The deduction of (iii) from (i) and (ii) is similar. $\square$

Definition 1.2. A bilinear form $\psi : V \times W \to F$ is *non-degenerate* if it satisfies the conditions of Theorem 1.1.

Equivalently, ψ is non-degenerate if and only if $\mathrm{rank}(\psi) = \dim V = \dim W$. Recall that the rank of ψ is the rank of any matrix representing it.

Remark 1.3. The set of bilinear forms $V \times V \to F$ under pointwise operations form a vector space over F. We may identify this space with $L(V, V^*)$ via $\psi \mapsto \psi_L$. The non-degenerate bilinear forms correspond to the isomorphisms from V to V^*.

2. Orthogonal complements

Let $\psi : V \times V \to F$ be a bilinear form. We assume that either ψ is symmetric, i.e. $\psi(u, v) = \psi(v, u)$ for all $u, v \in V$, or ψ is skew-symmetric, i.e. $\psi(u, v) = -\psi(v, u)$ for all $u, v \in V$. Then for $W \leq V$ we define the orthogonal subspace

$$W^{\perp} = \{v \in V : \psi(v, w) = 0 \text{ for all } w \in W\}.$$

The restriction of ψ to W (denoted $\psi|_W$ although we really mean $\psi|_{W\times W}$) is non-degenerate if and only if $W \cap W^{\perp} = \{0\}$.

Remark 2.1. It is possible for the restriction of a non-degenerate form to be degenerate. For example let $\psi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by $(u, v) \mapsto u^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$, and let $W = \langle \mathbf{e}_1 \rangle$. Then ψ is non-degenerate but $\psi|_W$ is degenerate. This is in contrast to the situation for positive definite forms: the restriction of a positive definite form is always positive definite.

Theorem 2.2. *Let $\psi : V \times V \to F$ be bilinear and either symmetric or skew-symmetric. Let $W \leq V$ be a subspace. Then* $\dim(W) + \dim(W^{\perp}) = \dim(V) + \dim(W \cap V^{\perp})$.

Proof: Let α be the restriction of ψ_R to W, i.e.

$$\begin{aligned} \alpha : W &\to V^* \\ w &\mapsto (v \mapsto \psi(v, w)). \end{aligned}$$

The rank-nullity theorem says

$$\dim(W) = \dim \operatorname{Ker}(\alpha) + \dim \operatorname{Im}(\alpha).$$

But $\operatorname{Ker}(\alpha) = \{w \in W : \psi(v, w) = 0 \text{ for all } v \in V\} = W \cap V^{\perp}$ and

$$\begin{aligned} \operatorname{Im}(\alpha)^{\circ} &= \{v \in V : \theta(v) = 0 \text{ for all } \theta \in \operatorname{Im}(\alpha)\} \\ &= \{v \in V : \alpha(w)(v) = 0 \text{ for all } w \in W\} \\ &= \{v \in V : \psi(v, w) = 0 \text{ for all } w \in W\} \\ &= W^{\perp}. \end{aligned}$$

Since for $U \leq V$ we have $\dim(U) + \dim(U^{\circ}) = \dim V$ it follows that

$$\dim(W) = \dim(W \cap V^{\perp}) + (\dim(V) - \dim(W^{\perp})).$$

□

Corollary 2.3. *Let $\psi : V \times V \to F$ be bilinear and either symmetric or skew-symmetric. Let $W \leq V$ be a subspace. Then*

$$\psi|_W \textit{ is non-degenerate } \iff V = W \oplus W^{\perp}.$$

Proof: "$\Rightarrow$" Since $\psi|_W$ is non-degenerate we have $W \cap W^{\perp} = \{0\}$. Therefore $W + W^{\perp}$ is a direct sum. Then $\dim(W \oplus W^{\perp}) = \dim(W) + \dim(W^{\perp}) \geq \dim(V)$ by Theorem 2.2. Hence $V = W \oplus W^{\perp}$.
"$\Leftarrow$" If $V = W \oplus W^{\perp}$ then $W \cap W^{\perp} = \{0\}$ and $\psi|_W$ is non-degenerate. □

We used a special case of Corollary 2.3 (with W a 1-dimensional subspace) in the proof that a symmetric bilinear form can be diagonalised.

3. Skew-symmetric forms

We assume that F is a field of characteristic not 2.

Definition 3.1. A bilinear form $\psi : V \times V \to F$ is *alternating* if $\psi(v,v) = 0$ for all $v \in V$.

Lemma 3.2. *Let $\psi : V \times V \to F$ be a bilinear form. Then ψ is alternating if and only if it is skew-symmetric.*

Proof: "$\Rightarrow$" By bilinearity

$$\psi(u+v, u+v) = \psi(u,u) + \psi(u,v) + \psi(v,u) + \psi(v,v).$$

Since ψ is alternating this reduces to $\psi(u,v) = -\psi(v,u)$, i.e. ψ is skew-symmetric.
"$\Leftarrow$" Since $\psi(v,v) = -\psi(v,v)$ (and $2 \neq 0$ in F) we get $\psi(v,v) = 0$. $\square$

Theorem 3.3. *Let V be a finite dimensional vector space over F and let $\psi : V \times V \to F$ be an alternating bilinear form. Then there exists a basis $\mathcal{B}$ for V such that*

$$[\psi]_{\mathcal{B}} = \left(\begin{array}{cccccc|c} 0 & 1 & & & & & \\ -1 & 0 & & & 0 & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & 0 \\ & & & & \ddots & & \\ & 0 & & & & 0 \quad 1 & \\ & & & & & -1 \quad 0 & \\ \hline & & & 0 & & & 0 \end{array}\right).$$

In particular the rank of ψ is even.

Proof: The proof is by induction on $\dim V$. If ψ is the zero form then we are done. Otherwise pick $v_1, v_2 \in V$ with $\psi(v_1, v_2) \neq 0$. Then ψ alternating implies v_1 and v_2 are linearly independent. Replacing v_2 by cv_2 for some non-zero $c \in F$ we may assume that $\psi(v_1, v_2) = 1$. Put $W = \langle v_1, v_2 \rangle$. Then $\psi|_W$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. By Corollary 2.3 (or a direct argument of the sort we used in the proof for symmetric forms) we have $V = W \oplus W^\perp$. Applying the induction hypothesis to $\psi|_{W^\perp}$ gives a basis $v_3, \ldots, v_n$ for $W^\perp$. Then $v_1, \ldots, v_n$ is the required basis for V. $\square$

Corollary 3.4. *If a finite dimensional vector space V admits a non-degenerate alternating bilinear form then $\dim V$ is even.*

4. Adjoints

Although we will meet adjoints in the section of the course on inner products, they can be defined more generally for non-degenerate bilinear forms. As before V will be a finite dimensional vector space over F.

Lemma 4.1. *Let ϕ and ψ be bilinear forms on V with ψ non-degenerate. Then there exists a unique $\alpha \in \mathrm{End}(V)$ such that*

$$\phi(v,w) = \psi(v, \alpha(w))$$

for all $v, w \in V$.

Proof: There are linear maps

$$\phi_R : V \to V^* \,; \quad w \mapsto (v \mapsto \phi(v, w))$$
$$\psi_R : V \to V^* \,; \quad w \mapsto (v \mapsto \psi(v, w)).$$

Since ψ is non-degenerate, ψ_R is an isomorphism. We put $\alpha = \psi_R^{-1} \circ \phi_R$. Then

$$\begin{aligned} & \psi_R \circ \alpha = \phi_R & \\ \implies \quad & \psi_R(\alpha(w))(v) = \phi_R(w)(v) & \text{for all } v, w \in V \\ \implies \quad & \psi(v, \alpha(w)) = \phi(v, w) & \text{for all } v, w \in V. \end{aligned}$$

Uniqueness: Suppose $\alpha_1, \alpha_2 \in \operatorname{End}(V)$ are solutions. Then $\psi(v, \alpha_1(w)) = \phi(v, w) = \psi(v, \alpha_2(w))$ for all $v, w \in V$. Then $\psi(v, \alpha_1(w) - \alpha_2(w)) = 0$ for all $v, w \in V$, and by non-degeneracy of ψ it follows that $\alpha_1 = \alpha_2$. □

Theorem 4.2. *Let $\psi : V \times V \to F$ be a non-degenerate bilinear form. For each $\alpha \in \operatorname{End}(V)$ there exists a unique $\alpha^* \in \operatorname{End}(V)$ such that*

$$\psi(\alpha(v), w) = \psi(v, \alpha^*(w))$$

for all $v, w \in V$. We call α^* the adjoint of α.

Proof: Define $\phi : V \times V \to F$ by $(v, w) \mapsto \psi(\alpha(v), w)$. Then ϕ is bilinear and Lemma 4.1 constructs α^*. □

Remark 4.3. If ψ is non-degenerate then $\psi_R : V \to V^*$ is an isomorphism. If we identify V and V^* via this map then the adjoint α^* works out as being the same as the dual map (as defined in the section on dual spaces, and also denoted α^*.)