Michaelmas Term 2006 J. Saxl

Linear Algebra: Example Sheet 4

The first ten questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be a bit harder; they are intended to be attempted only after completion of the first part.

1. The square matrices A and B over the field F are congruent if $B = P^t AP$ for some invertible matrix P over F. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) [Try to get away with the minimum calculation.]
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.
$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$
If A is the matrix of the first of these (say), find a non-singular matrix P such that $P^t AP$ is diagonal with entries ± 1.

3. Show that the function $\psi(A, B) = \mathrm{tr}(AB^t)$ is a symmetric positive-definite bilinear form on the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices.
Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form on $M_n(\mathbb{R})$. Find its rank and signature.

4. Find the rank and signature of the form on $\mathbb{R}^n$ with matrix
$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

5. Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite. *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis obtained by applying the Gram-Schmidt process to the standard basis.

6. Let $W \leq V$ with V an inner product space. Show that the orthogonal projection onto W is a self-adjoint projection. Conversely show that any self-adjoint projection is orthogonal projection onto its image.

7. Let S be a real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. An endomorphism α of a finite-dimensional inner product space V is *positive semi-definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in V$. Prove that a positive semi-definite endomorphism has a unique positive semi-definite square root.

9. Let V be a complex inner product space, let α be an endomorphism on V. Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^*. Deduce that there is an othonormal basis of eigenvectors of α.

10. Find a linear transformation which reduces the pair of real quadratic forms
$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$
to the forms
$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$
for some $\lambda, \mu, \nu \in \mathbb{R}$ (which you will inevitably determine).
Does their exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2$, $2xy$ simultaneously to diagonal forms?

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11. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p+q$ and signature $p-q$. Show that $p \leq t$ and $q \leq u$.

12. Suppose that Q is a non-singular quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \leq V$ with $\dim U = m$. What is the signature of Q? Establish the following.
(i) We can write $V = U \oplus W$ with Q also vanishing on W.
(ii) There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \cdots x_{2m-1}x_{2m}$.

13. The principal minors of an $n \times n$ matrix are the determinants of the n square submatrices in its top left corner, one of each size up to n. Prove that a real symmetric matrix is positive definite if and only if all its principal minors are positive.

14. Suppose that α is an orthogonal endomorphism on the finite-dimensional real inner product space V. Prove that V can be decomposed into a direct sum of mutually orthogonal α-invariant subspaces of dimension 1 or 2. Determine the possible matrices of α with respect to orthonormal bases in the cases where V has dimension 1 or dimension 2.

15. (i) Show that $O_n(\mathbb{R})$, the group of all real orthogonal $n \times n$ matrices, has a normal subgroup $SO_n(\mathbb{R})$ consisting of the real orthogonal matrices of determinant $+1$.
(ii) Show that the centre of $O_n(\mathbb{R})$ is $\{\pm I\}$.
(iii) Show that $O_n(\mathbb{R})$ is the direct product of $SO_n(\mathbb{R})$ and the centre $\{\pm I\}$ if and only if n is odd.
(iv) Show that if n is even then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

16. Let V be a complex inner product space, let α be an invertible endomorphism on V. By considering $\alpha^*\alpha$, show that α can be factored as $\beta\gamma$ with β unitary and γ positive.

17. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define
$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt\,.$$
Show that $\langle\ ,\ \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by
$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$
is self-adjoint. What are the eigenvalues of α?
Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1-t^2)^k$. Prove the following.
(i) For $i \neq j$, $\langle s_i, s_j \rangle = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

18. Let (r, ϕ, θ) be the standard spherical polar coordinates in $\mathbb{R}^3$. Let $\mathbf{u}_r$, $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ represent the unit vectors in the direction of increasing r, ϕ and θ respectively.
Write down the matrix giving $\mathbf{u}_r$, $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ in terms of the standard basis $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ for $\mathbb{R}^3$.
What is the inverse of this matrix and why?
What real value must be an eigenvalue of the matrix and why?
(You could check by explicit calculation of the characteristic polynomial that you are right.)

19. Let a_1, a_2, ... , a_n be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

20. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix with $|a_{ij}| \leq k$, then
$$|\det A| \leq k^n n^{n/2}\,.$$

Comments, corrections and queries can be sent to me at `saxl@dpmms.cam.ac.uk`.