Michaelmas Term 2006 J. Saxl

Linear Algebra: Example Sheet 1

The first twelve questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they are intended to be attempted only after completion of the first part.

1. Let $F(\mathbb{R})$ be the vector space (sometimes denoted $\mathbb{R}^{\mathbb{R}}$) of all functions $f:\mathbb{R}\to\mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $F(\mathbb{R})$? [You should attempt these questions, but may not want to write out all parts in much detail.]
(a) The set C of continuous functions.
(b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.
(c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
(d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
(e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2-3)\dot{x}(t) + t^4 x(t) = 0$.
(f) The set of solutions of $\ddot{x}(t) + (t^2-3)\dot{x}(t) + t^4 x(t) = \sin t$.
(g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that T and U are subspaces of the vector space V. Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.

3. Let T, U, W be subspaces of V.
(i) Give explicit counter-examples to the following statements:
(a) $T + (U \cap W) = (T+U) \cap (T+W)$; (b) $(T+U) \cap W = (T \cap W) + (U \cap W)$.
(ii) Show in both (a) and (b) that the equality can be replaced by a valid inclusion of one side in the other.
(iii) Show that if $T \leq W$, then $(T+U) \cap W = T + (U \cap W)$.

4. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for V. Which of the following are also bases?
(a) $\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \ldots, \mathbf{e}_{n-1}+\mathbf{e}_n, \mathbf{e}_n$;
(b) $\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_2+\mathbf{e}_3, \ldots, \mathbf{e}_{n-1}+\mathbf{e}_n, \mathbf{e}_n+\mathbf{e}_1$;
(c) $\mathbf{e}_1-\mathbf{e}_2, \mathbf{e}_2-\mathbf{e}_3, \ldots, \mathbf{e}_{n-1}-\mathbf{e}_n, \mathbf{e}_n-\mathbf{e}_1$;
(d) $\mathbf{e}_1-\mathbf{e}_n, \mathbf{e}_2+\mathbf{e}_{n-1}, \ldots, \mathbf{e}_n+(-1)^n\mathbf{e}_1$.

5. Let P denote the space of all polynomial functions $\mathbb{R}\to\mathbb{R}$. Which of the following define linear maps $P \to P$? [You should attempt these questions, but may not want to write out all parts in much detail.]
(a) $D(p)(t) = p'(t)$.
(b) $S(p)(t) = p(t^2+1)$.
(c) $T(p)(t) = p(t)^2 + 1$.
(d) $E(p)(t) = p(e^t)$.
(e) $J(p)(t) = \int_0^t p(s)\,ds$.
(f) $K(p)(t) = 1 + \int_0^t p(s)\,ds$.
(g) $L(p)(t) = p(0) + \int_0^t p(s)\,ds$.
(h) $M(p)(t) = p(t^2) - tp(t)$.

6. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R}\to\mathbb{R}$, and $C[a,b]$ denotes the space of continuous functions defined on the closed interval $[a,b]$.)
(a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1+x_2+x_3+x_4+x_5 = 0\}$.
(b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0,1]$, $W = C[-1,1]$.
(d) $V = C[0,1]$, $W = \{f \in C[0,1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4$, $W = C[0,1]$.
(g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

7. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ \ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ \ x_2 = x_3 = x_4\}.$$

Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}\,.$$

8. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

9. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the base $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

10. If $U_1, \ldots, U_k$ are subspaces of a vector space V, show that the following conditions are equivalent:
(i) every element of $\sum_i U_i$ can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$;
(ii) if B_i is a basis of U_i, the union of the B_i is a basis for $\sum_i U_i$;
(iii) for each j, $\quad U_j \cap \sum_{i \neq j} U_i = \{\mathbf{0}\}$.

Show that these conditions are **not** equivalent to
(iv) for each $i \neq j$, $\quad U_i \cap U_j = \{\mathbf{0}\}$.

11. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\alpha \in L(V, W) : \alpha(\mathbf{x}) \in Z \text{ for all } \mathbf{x} \in Y\}$ is a subspace of the space $L(V, W)$ of all linear maps from V to W. What is the dimension of R?

12. Let V be a vector space over F, let W be a subspace. Show that there is a natural way in which the quotient group V/W is a vector space over F. Show that if the dimension of V is finite, then so is the dimension of V/W, and

$$\dim V = \dim W + \dim V/W.$$

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13. X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample. [Look at $\mathbb{R}^3$.]

14. Let U be a proper subspace of the finite-dimensional vector space V. Find a basis for V containing no element of U.

15. (Another version of the Steinitz Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite-dimensional vector space have the same size.

16. Show that any two subspaces of the same dimension in a finite-dimensional vector space have a common complementary subspace. [You may wish to consider first the case where the subspaces have dimension 1 less than the space.]

17. Let F_p be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over F_p. How many vectors are there in V? How many bases? How many automorphisms does V have? How many k-dimensional subspaces are there in V?

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