Michaelmas Term 2004 J. Saxl

Linear Algebra: Example Sheet 3

The first eleven questions cover the relevant part of the course and should ensure good understanding. The remaining questions on the next page may or may not be harder; they should only be attempted after completion of the first part.

1. Let V be a vector space, let $\pi_1, \pi_2, \ldots, \pi_k$ be endomorphisms of V such that $\iota = \pi_1 + \cdots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.

2. Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

3. Show that none of the following matrices are conjugate:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Is the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
conjugate to any of them? If so, which?

4. Show that for 3×3 complex matrices, their Jordan normal form can be deduced from their characteristic and minimal polynomials. Give an example to show that this is not so for 4×4 complex matrices. Give an example, for any $a \leq n$, of an $n \times n$ complex matrix with minimal polynomial t^a.

5. Find a basis with respect to which $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ is in Jordan normal form. Hence compute $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

6. Let α be an endomorphism of the finite dimensional complex vector space V, with characteristic polynomial $\chi_\alpha(t) = (-t)^n + c_{n-1}t^{n-1} + \cdots + c_0$. Show that $c_0 = \det(\alpha)$ and $(-1)^{n-1}c_{n-1} = \mathrm{tr}(\alpha)$.

7. Let α be an endomorphism of the finite-dimensional complex vector space V, and assume that α is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of α^{-1} in terms of those of α.

8. Let V be a complex vector space of dimension n and let α be an endomorphism of V with $\alpha^{n-1} \neq 0$ but $\alpha^n = 0$. Show that there is a vector $\mathbf{x} \in V$ for which $\mathbf{x},\ \alpha(\mathbf{x}),\ \alpha^2(\mathbf{x}),\ \ldots,\ \alpha^{n-1}(\mathbf{x})$ is a basis for V. Give the matrix of α relative to this basis.
Let $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to the base? What is the minimal polynomial for α? What are the eigenvalues and eigenvectors?
Show that if an endomorphism β of V commutes with α then $\beta = p(\alpha)$ for some polynomial $p(t)$.
[It may help to consider $\beta(\mathbf{x})$.]

9. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ form a basis for $\mathbb{R}^3$. Find the dual basis for $(\mathbb{R}^3)^*$.

10. Let V be a 4-dimensional vector space over $\mathbb{R}$, and let $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ be the basis of V^* dual to the basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ for V. Determine, in terms of the ξ_i, the bases dual to each of the following:
(a) $\{\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_4, \mathbf{x}_3\}$;
(b) $\{\mathbf{x}_1, 2\mathbf{x}_2, \frac{1}{2}\mathbf{x}_3, \mathbf{x}_4\}$;
(c) $\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4\}$;
(d) $\{\mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 + \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3 + \mathbf{x}_2 - \mathbf{x}_1\}$.

11. Show that if $\mathbf{x} \neq \mathbf{y}$ are vectors in the finite dimensional vector space V, then there is a linear functional $\theta \in V^*$ such that $\theta(\mathbf{x}) \neq \theta(\mathbf{y})$.

---

12. Let V be a vector space of all complex sequences (z_n) which satisfy the difference equation $z_{n+2} = 3z_{n+1} - 2z_n$ for $n = 1, 2, \ldots$. Write down an obvious basis for V and hence determine its dimension. Show that the "shift" operator which sends a sequence $(z_1, z_2, z_3, \ldots)$ to $(z_2, z_3, z_4, \ldots)$ is an endomorphism on V. Find the matrix which represents this map with respect to your basis. Show that there is a basis for V with respect to which the map is represented by a diagonal matrix.

    What happens if we replace the difference equation by $z_{n+2} = 2z_{n+1} - z_n$?

13. Let A be a square complex matrix of finite order - that is, $A^m = I$ for some m. Show that A can be diagonalized.

14. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

15. (Another proof of the Diagonizability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonizable.

16. Let V be a vector space of finite dimension over a field F. Let α be an endomorphism of V and let U be a proper α-invariant subspace of V (so $\alpha(U) \leq U$). Consider a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of V containing a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of U. Write $\overline{V} = V/U$, $\overline{\mathbf{v}} = \mathbf{v} + U$, and define $\overline{\alpha} \in L(\overline{V})$ by $\overline{\alpha}(\overline{\mathbf{v}}) = \overline{\alpha(\mathbf{v})}$. Show that the matrix of α with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is $A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$, with B the matrix of the restriction α_U of α to U with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and C the matrix of $\overline{\alpha}$ with respect to $\overline{\mathbf{v}_{k+1}}, \ldots, \overline{\mathbf{v}_n}$. Deduce that $\chi_\alpha = \chi_{\alpha_U}\chi_{\overline{\alpha}}$.

17. (Another proof of the Cayley Hamilton Theorem.) Let V be a vector space of finite dimension over a field F and let α be an endomorphism of V. If U is a proper α-invariant subspace of V, use the previous question and induction to show that $\chi_\alpha(\alpha) = 0$. If no such subspace exists, show that there exists a basis $\mathbf{v}, \alpha(\mathbf{v}), \ldots \alpha^{n-1}(\mathbf{v})$ of V. Show that α has matrix

$$\begin{pmatrix} 0 & & & -a_o \\ 1 & \ddots & & -a_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -a_{n-1} \end{pmatrix}$$

    with respect to this basis, for suitable $a_i \in F$. By expanding in the last column or otherwise, show that $(-1)^n\chi_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Show that $\chi_\alpha(\alpha)(\mathbf{v}) = 0$, and deduce that $\chi_\alpha(\alpha)$ is 0 on V.

18. Let θ and ϕ be linear functionals on V with the property that $\theta(\mathbf{x}) = 0$ if and only if $\phi(\mathbf{x}) = 0$. Show that θ and ϕ are scalar multiples of each other.

19. Suppose that V is finite dimensional. Let $A, B \leq V$. Prove that $A \leq B$ if and only if $A^o \geq B^o$. Show that $A = V$ if and only if $A^o = \{\mathbf{0}\}$. Deduce that a subset $F \subset V^*$ of the dual space spans V^* just when $f(\mathbf{v}) = 0$ for all $f \in F$ implies $\mathbf{v} = \mathbf{0}$.

20. Show that the dual of the space P of real polynomials is isomorphic to the space $\mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \to \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

    In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \to P$:
    (a) The map D defined by $D(p)(t) = p'(t)$.
    (b) The map S defined by $S(p)(t) = p(t^2)$.
    (c) The map E defined by $E(p)(t) = p(t - 1)$.
    (d) The composite DS.
    (e) The composite SD.

    Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

Comments, corrections and queries can be sent to me at saxl@dpmms.cam.ac.uk.