Linear Algebra: Example Sheet 2

The first twelve questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part.

1. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Set $r_k = \mathrm{r}(\alpha^k)$. Show that $r_k \geq r_{k+1}$, and that $(r_k - r_{k+1}) \geq (r_{k+1} - r_{k+2})$. [Consider the restriction of α to $\mathrm{Im}(\alpha^k)$.]
   (ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

2. Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorizes as $A = BC$ with $B \in M_{m\times r}$ and $C \in M_{r\times n}$.
   Using the fact that $(BC)^t = C^t B^t$, deduce that the (column) rank of A^t equals r.

3. For what values of a and b does the system of simultaneous linear equations
$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$
   have (i) a unique solution, (ii) no solution, (iii) many solutions?

4. Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. [Note that the sum of all columns of A has all entries equal.]

5. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $(2n \times 2n)$ matrix
$$C = \begin{pmatrix} I & B \\ -A & O \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
   by elementary row operations. By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

6. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are conjugate when regarded as matrices over $\mathbb{C}$, then they are conjugate as matrices over $\mathbb{R}$.

7. Let V be a non-trivial vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \iota$.

8. Compute the characteristic polynomials of the matrices
$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
   Which of the matrices are diagonalizable over $\mathbb{C}$? Which over $\mathbb{R}$?

9. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$
   The second and third matrices commute; find a basis with respect to which they are both diagonal.

10. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Show that the characteristic polynomial is $t^3 - 2t + 1$. Hence compute $A^7 - 2A^5 + 2A^4 - 2A^2 + 2A + I$ and A^{-1}.

11. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\ker(\alpha - \lambda I)$ and $\ker(\alpha^2 - \lambda^2 I)$ necessarily the same?

12. Show that an endomorphism $\alpha : V \to V$ of a finite dimensional complex vector space V has 0 as only eigenvalue if and only if it is *nilpotent*, that is, $\alpha^k = 0$ for some natural number k. Show that the minimum such k is at most $\dim(V)$. What can you say if the only eigenvalue of α is 1?

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13. Let A be an $n \times m$ matrix. Prove that if B is an $m \times n$ matrix then

$$\mathrm{r}(AB) \le \min(\mathrm{r}(A), \mathrm{r}(B)).$$

At the start of each year the jovial and popular Dean of Muddling (pronounced Chumly) College organizes m parties for the n students of the College. Each student is invited to exactly k parties, and every two students are invited to exactly one party in common. Naturally $k \ge 2$. Let $P = (p_{ij})$ be the $n \times m$ matrix defined by

$$p_{ij} = \begin{cases} 1 & \text{if student i is invited to party j} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the matrix PP^t and find its rank. Deduce that $m \ge n$. (Fisher's inequality according to TWK.)
After the Master's cat has been found dyed green, maroon and purple on successive nights, the other fellows insist that next year $k = 1$. Why does the proof above now fail, and what will, in fact, happen next year? (The answer required is mathematical rather than sociological in nature.)

14. Let A, B be $n \times n$ matrices, where $n \ge 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)\,, \quad (ii)\ \det(\mathrm{adj}A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the matrix $\mathrm{adj}A$ is $\mathrm{r}(\mathrm{adj}(A)) = \begin{cases} n & \text{if } \mathrm{r}(A) = n; \\ 1 & \text{if } \mathrm{r}(A) = n-1; \\ 0 & \text{if } \mathrm{r}(A) \le n-2. \end{cases}$

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \ne 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

17. Let V denote the space of all infinitely differentiable real functions (with pointwise operations, as usual), and let α be the differentiation endomorphism $f \mapsto f'$.
(a) By considering $f(t) = e^{\lambda t}$, show that every real number λ is an eigenvalue of α (that is, $\alpha - \lambda\iota$ is not injective for any λ). Show also that the kernel of $\alpha - \lambda\iota$ has dimension 1.
(b) Show that $\alpha - \lambda\iota$ is surjective for every λ. [Given $f \in V$, consider $g(t) = e^{\lambda t} \int_0^t e^{-\lambda s} f(s)\, ds$.]

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