Michaelmas Term 2004 J. Saxl

Linear Algebra: Example Sheet 1

The first eleven questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part.

1. Let $F(\mathbb{R})$ be the vector space (sometimes denoted $\mathbb{R}^{\mathbb{R}}$) of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $F(\mathbb{R})$? [You should attempt these questions, but may not want to write out all parts in much detail.]
   (a) The set C of continuous functions.
   (b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.
   (c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
   (d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
   (e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.
   (f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.
   (g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
   (h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that T and U are subspaces of the vector space V. Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.

3. Let T, U, W be subspaces of V.
   (i) Give explicit counter-examples to the following statements:
   (a) $T + (U \cap W) = (T + U) \cap (T + W)$; (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.
   (ii) Show in both (a) and (b) that the equality can be replaced by a valid inclusion of one side in the other.
   (iii) Show that if $T \leq W$, then $(T + U) \cap W = T + (U \cap W)$.

4. Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V. Which of the following are also bases?
   (a) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n\}$.
   (b) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1\}$.
   (c) $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1\}$.
   (d) $\{\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1\}$.

5. Let P denote the space of all polynomial functions $\mathbb{R} \to \mathbb{R}$. Which of the following define linear maps $P \to P$?
   (a) $D(p)(t) = p'(t)$.
   (b) $S(p)(t) = p(t^2 + 1)$.
   (c) $T(p)(t) = p(t)^2 + 1$.
   (d) $E(p)(t) = p(e^t)$.
   (e) $J(p)(t) = \int_0^t p(s)\, ds$.
   (f) $K(p)(t) = 1 + \int_0^t p(s)\, ds$.
   (g) $L(p)(t) = p(0) + \int_0^t p(s)\, ds$.
   (h) $M(p)(t) = p(t^2) - tp(t)$.

6. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.)
   (a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
   (b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
   (c) $V = C[0, 1]$, $W = C[-1, 1]$.
   (d) $V = C[0, 1]$, $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable }\}$.
   (e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
   (f) $V = \mathbb{R}^4$, $W = C[0, 1]$.
   (g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

7. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}.$$

Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

8. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

9. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the base $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

10. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\theta \in L(V, W) : \theta(\mathbf{x}) \in Z \text{ for all } \mathbf{x} \in Y\}$ is a subspace of the space $L(V, W)$ of all linear maps from V to W. What is the dimension of R?

11. If $U_1, \ldots, U_k$ are subspaces of a vector space V, show that the following conditions are equivalent.
(i) $\dim \sum_i U_i = \sum_i \dim U_i$;
(ii) every element of $\sum_i U_i$ can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$;
(iii) For each j, $\quad U_j \cap \sum_{i \neq j} U_i = \{\mathbf{0}\}$.
Show that these conditions are **not** equivalent to
(iv) For each $i \neq j$, $\quad U_i \cap U_j = \{\mathbf{0}\}$.

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12. Let U be a proper subspace of the finite-dimensional vector space V. Find a basis for V containing no element of U.

13. X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample. [Look at $\mathbb{R}^3$.]

14. (Another version of the Steinitz Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite-dimensional vector space have the same size.

15. Show that any two subspaces of the same dimension in a finite-dimensional vector space have a common complementary subspace. [You may wish to consider first the case where the subspaces have dimension 1 less than the space.]

16. Let V be a vector space over F, let W be a subspace. Show that there is a natural way in which the quotient group V/W is a vector space over F. Show that if the dimension of V is finite, then the dimension of V/W satisfies

$$\dim V/W = \dim V - \dim W.$$

17. Let F_p be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over F_p. How many vectors are there in V? How many bases? How many automorphisms does V have? How many k-subspaces are there in V?

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