

**IB Groups, Rings, and Modules // Example Sheet 4**

1. Let  $M$  be a module over a ring  $R$ , and let  $N$  be a submodule of  $M$ .
  - (i) Show that if  $M$  is finitely generated then so is  $M/N$ .
  - (ii) Show that if  $N$  and  $M/N$  are finitely generated then so is  $M$ .
  - (iii) Show that if  $M/N$  is free, then  $M \cong N \oplus M/N$ .
2. We say that an  $R$ -module satisfies condition  $(N)$  if any submodule is finitely generated. Show that this condition is equivalent to condition  $(ACC)$ : every increasing chain of submodules terminates. Let  $R$  be a Noetherian ring. Show that the  $R$ -module  $R^n$  satisfies condition  $(N)$ , and hence that any finitely generated  $R$ -module satisfies condition  $(N)$ .
3. Let  $M$  be a module over an integral domain  $R$ . An element  $m \in M$  is a *torsion element* if  $rm = 0$  for some non-zero  $r \in R$ .
  - (i) Show that the set  $T$  of all torsion elements in  $M$  is a submodule of  $M$ , and that the quotient  $M/T$  is *torsion-free*—that is, contains no non-zero torsion elements.
  - (ii) What are the torsion elements in the  $\mathbb{Z}$ -module  $\mathbb{Q}/\mathbb{Z}$ ? In  $\mathbb{R}/\mathbb{Z}$ ? In  $\mathbb{R}/\mathbb{Q}$ ?
  - (iii) Is the  $\mathbb{Z}$ -module  $\mathbb{Q}$  torsion-free? Is it free? Is it finitely generated?

4. Use elementary operations to put  $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix} \in M_{3,3}(\mathbb{Z})$  into Smith normal form  $D$ .

Check your result using minors. Explain how to find invertible matrices  $P, Q$  for which  $D = QAP$ .

5. Work out the Smith normal form of the matrices

$$\begin{pmatrix} 2X-1 & X & X-1 & 1 \\ X & 0 & 1 & 0 \\ 0 & 1 & X & X \\ 1 & X^2 & 0 & 2X-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X^2+2X & 0 & 0 & 0 \\ 0 & X^2+3X+2 & 0 & 0 \\ 0 & 0 & X^3+2X^2 & 0 \\ 0 & 0 & 0 & X^4+X^3 \end{pmatrix}$$

over  $\mathbb{R}[X]$ .

6. How many abelian groups are there of order 6? Of order 60? Of order 6000?
7. Let  $G$  be the abelian group with generators  $a, b, c$ , and relations  $6a + 10b = 0$ ,  $6a + 15c = 0$ ,  $10b + 15c = 0$ . (That is,  $G$  is the free abelian group on generators  $a, b, c$  quotiented by the subgroup generated by the elements  $6a + 10b$ ,  $6a + 15c$ ,  $10b + 15c$ ). Determine the structure of  $G$  as a direct sum of cyclic groups.
8. Prove that a finitely-generated abelian group  $G$  is finite if and only if  $G/pG = 0$  for some prime  $p$ . Give a non-trivial abelian group  $G$  such that  $G/pG = 0$  for all primes  $p$ .
9. Let  $A$  be a complex matrix with characteristic polynomial  $(X+1)^6(X-2)^3$  and minimal polynomial  $(X+1)^3(X-2)^2$ . Write down the possible Jordan normal forms for  $A$ . What are the invariant factors of the corresponding  $\mathbb{C}[X]$ -modules?
10. Find a  $2 \times 2$  matrix over  $\mathbb{Z}[X]$  that is not equivalent to a diagonal matrix.
11. Let  $M$  be a finitely-generated module over a Noetherian ring  $R$ , and let  $f$  be an  $R$ -module homomorphism from  $M$  to itself. Does  $f$  injective imply  $f$  surjective? Does  $f$  surjective imply  $f$  injective?

### Additional Questions

12. A real  $n \times n$  matrix  $A$  satisfies the equation  $A^2 + I = 0$ . Show that  $n$  is even and  $A$  is similar to a block matrix  $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$  with each block an  $m \times m$  matrix (where  $n = 2m$ ).
13. Show that a complex number  $\alpha$  is an algebraic integer if and only if the additive group of the ring  $\mathbb{Z}[\alpha]$  is finitely generated (i.e.  $\mathbb{Z}[\alpha]$  is a finitely generated  $\mathbb{Z}$ -module). Furthermore if  $\alpha$  and  $\beta$  are algebraic integers show that the subring  $\mathbb{Z}[\alpha, \beta]$  of  $\mathbb{C}$  generated by  $\alpha$  and  $\beta$  also has a finitely generated additive group and deduce that  $\alpha - \beta$  and  $\alpha\beta$  are algebraic integers.  
Show that the algebraic integers form a subring of  $\mathbb{C}$ .
14. Show that the ring  $C^\infty([-1, 1])$  of all infinitely differentiable functions  $[-1, 1] \rightarrow \mathbb{R}$  (with pointwise operations) is not Noetherian.
15. What is the rational canonical form of a matrix?  
Show that the group  $GL_2(\mathbb{F}_2)$  of non-singular  $2 \times 2$  matrices over the field  $\mathbb{F}_2$  with 2 elements has three conjugacy classes of elements.  
Show that the group  $GL_3(\mathbb{F}_2)$  of non-singular  $3 \times 3$  matrices over the field  $\mathbb{F}_2$  has six conjugacy classes of elements, corresponding to minimal polynomials  $X + 1, (X + 1)^2, (X + 1)^3, X^3 + 1, X^3 + X^2 + 1, X^3 + X + 1$ , one each of elements of orders 1, 2, 3 and 4, and two of elements of order 7.
16. Let  $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1) = \{0, 1, \omega, \omega + 1\}$ , a field with four elements.  
Show that the group  $SL_2(\mathbb{F}_4)$  of  $2 \times 2$  matrices of determinant 1 over  $\mathbb{F}_4$  has five conjugacy classes of elements, corresponding to minimal polynomials  $x + 1, (x + 1)^2, (x + \omega)(x + \omega^2), x^2 + \omega x + 1$  and  $x^2 + \omega^2 x + 1$ .  
Show that the corresponding elements have orders 1, 2, 3, 5 and 5, respectively.

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