Lent Term 2021 T.A. Fisher

Groups Rings and Modules: Example Sheet 4 of 4

All rings in this course are commutative with a 1.

1. Let M be a module over a ring R, and let N be a submodule of M.
   (i) Show that if N and M/N are finitely generated then so is M.
   (ii) Show that if M/N is free, then $M \cong N \oplus M/N$.

2. We say that an R-module satisfies condition (N) if any submodule is finitely generated. Show that this condition is equivalent to condition (ACC): every increasing chain of submodules terminates.

   Let R be a Noetherian ring. By considering first the case of a cyclic R-module, or otherwise, show that any finitely generated R-module satisfies condition (N).

3. Let M be a module over an integral domain R. We say that $m \in M$ is a *torsion element* if $rm = 0$ for some non-zero $r \in R$.
   (i) Show that the set T of all torsion elements in M is a submodule of M, and that the quotient M/T is *torsion-free*—that is, contains no non-zero torsion elements.
   (ii) What are the torsion elements in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$? In $\mathbb{R}/\mathbb{Z}$? In $\mathbb{R}/\mathbb{Q}$?
   (iii) Is the $\mathbb{Z}$-module $\mathbb{Q}$ torsion-free? Is it free? Is it finitely generated?

4. Use elementary operations to put the integer matrix $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$ into Smith normal form D. Check your result using minors. Explain how to find invertible matrices P, Q for which $D = QAP$.

5. Work out the Smith normal form of the matrices over $\mathbb{R}[X]$:

$$\begin{pmatrix} 2X-1 & X & X-1 & 1 \\ X & 0 & 1 & 0 \\ 0 & 1 & X & X \\ 1 & X^2 & 0 & 2X-2 \end{pmatrix} \text{ and } \begin{pmatrix} X^2+2X & 0 & 0 & 0 \\ 0 & X^2+3X+2 & 0 & 0 \\ 0 & 0 & X^3+2X^2 & 0 \\ 0 & 0 & 0 & X^4+X^3 \end{pmatrix}$$

6. How many abelian groups are there of order 6? Of order 60? Of order 6000?

7. Let G be the abelian group with generators a, b, c, and relations $6a+10b = 0$, $6a+15c = 0$, $10b + 15c = 0$. (That is, G is the free abelian group on generators a, b, c quotiented by the subgroup generated by the elements $6a + 10b,\ 6a + 15c,\ 10b + 15c$.) Determine the structure of G as a direct sum of cyclic groups.

8. Prove that a finitely generated abelian group G is finite if and only if $G/pG = 0$ for some prime p. Give an example of a non-trivial abelian group G such that $G/pG = 0$ for all primes p.

9. Let A be a complex matrix with characteristic polynomial $(X+1)^6(X-2)^3$ and minimal polynomial $(X+1)^3(X-2)^2$. Write down the possible Jordan normal forms for A. What are the invariant factors of the corresponding $\mathbb{C}[X]$-modules?

10. Find a 2×2 matrix over $\mathbb{Z}[X]$ that is not equivalent to a diagonal matrix. Find also a finitely generated module over $\mathbb{Z}[X]$ that is not isomorphic to a direct sum of cyclic modules.

11. Let M be a finitely generated module over a Noetherian ring R, and let f be an R-module homomorphism from M to itself. Does f injective imply f surjective? Does f surjective imply f injective? What happens if R is not Noetherian?

Further Questions

12. A real $n \times n$ matrix A satisfies the equation $A^2 + I = 0$. Show that n is even and A is similar to a block matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ with each block an $m \times m$ matrix (where $n = 2m$).

13. Show that a complex number α is an algebraic integer if and only if the additive group of the ring $\mathbb{Z}[\alpha]$ is finitely generated (i.e. $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module). Furthermore if α and β are algebraic integers show that the subring $\mathbb{Z}[\alpha, \beta]$ of $\mathbb{C}$ generated by α and β also has a finitely generated additive group and deduce that $\alpha - \beta$ and $\alpha\beta$ are algebraic integers.

    Show that the algebraic integers form a subring of $\mathbb{C}$.

14. Show that the ring $C^\infty([-1, 1])$ of all infinitely differentiable functions $[-1, 1] \to \mathbb{R}$ (with pointwise operations) is not Noetherian.

15. What is the rational canonical form of a matrix?

    Show that the group $\mathrm{GL}_2(\mathbb{F}_2)$ of non-singular 2×2 matrices over the field $\mathbb{F}_2$ of 2 elements has three conjugacy classes of elements.

    Show that the group $\mathrm{GL}_3(\mathbb{F}_2)$ of non-singular 3×3 matrices over $\mathbb{F}_2$ has six conjugacy classes of elements, corresponding to minimal polynomials $X + 1$, $(X + 1)^2$, $(X + 1)^3$, $X^3 + 1$, $X^3 + X^2 + 1$, $X^3 + X + 1$, one each of elements of orders $1, 2, 3$ and 4, and two of elements of order 7.

16. Let $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1) = \{0, 1, \omega, \omega + 1\}$, a field with four elements.

    Show that the group $\mathrm{SL}_2(\mathbb{F}_4)$ of 2×2 matrices of determinant 1 over $\mathbb{F}_4$ has five conjugacy classes of elements, corresponding to minimal polynomials $X + 1$, $(X + 1)^2$, $(X + \omega)(X + \omega^2)$, $X^2 + \omega X + 1$ and $X^2 + \omega^2 X + 1$.

    Show that the corresponding elements have orders $1, 2, 3, 5$ and 5, respectively.