IB Groups, Rings and Modules: Example Sheet 2

All rings in this course are commutative with a multiplicative identity.

1. Let $\omega = \frac{1}{2}(1+\sqrt{-3})$, let $R = \{a + b\omega :\ a, b \in \mathbb{Z}\}$, and let $F = \{a + b\omega :\ a, b \in \mathbb{Q}\}$. Show that R is a subring of $\mathbb{C}$, and that F is a subfield of $\mathbb{C}$. What are the units of R?

2. *An element r of a ring R is nilpotent if $r^n = 0$ for some n.*
(i) What are the nilpotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
(ii) Show that if r is nilpotent then r is not a unit, but $1 + r$ and $1 - r$ are units.
(iii) Show that the nilpotent elements form an ideal N in R. What are the nilpotent elements in the quotient ring R/N?

3. Let r be an element of a ring R. Show that, in the polynomial ring $R[X]$, the polynomial $1+rX$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

4. Show that if I and J are ideals in the ring R, then so is $I \cap J$, and the quotient $R/(I \cap J)$ is isomorphic to a subring of the product $R/I \times R/J$.

5. Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be ideals in a ring R. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.

6. Write down a prime ideal in $\mathbb{Z}^2$ that is not maximal. Explain why, in a finite ring, all prime ideals are maximal.

7. Explain why, for p a prime, there is a unique ring of order p. How many rings are there of order 4?

8. Let R be an integral domain and F be its field of fractions. Suppose that $\phi : R \to K$ is an injective ring homomorphism from R to a field K. Show that ϕ extends to an injective homomorphism $\Phi : F \to K$ from F to K. What happens if we do not assume that ϕ is injective?

9. Let R be any ring. Show that the ring $R[X]$ is a principal ideal domain if and only if R is a field.

10. *An element r of a ring R is idempotent if $r^2 = r$.*
(i) What are the idempotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
(ii) Show that if r is idempotent then so is $r' = 1 - r$, and $rr' = 0$. Show also that the ideal (r) is naturally a ring, and that R is isomorphic to $(r) \times (r')$.

11. (i) Show that the set $\mathbb{P}(S)$ of all subsets of a given set S is a ring with respect to the operations of symmetric difference and intersection. Describe the principal ideals in this ring. Show that the ideal (A, B) generated by elements A, B is in fact principal.
(ii) A ring R is called *Boolean* if every element of R is idempotent. Prove that every finite Boolean ring is isomorphic to a power-set ring $\mathbb{P}(S)$ for some set S. Give an example to show that this need not remain true for infinite Boolean rings.

Additional Questions

12. Is every abelian group the additive group of some ring?

13. Let I be an ideal of the ring R and $P_1, \ldots, P_n$ be prime ideals of R. Show that if $I \subset \bigcup_{i=1}^{n} P_i$, then $I \subset P_i$ for some i.

14. A sequence $\{a_n\}$ of rational numbers is a *Cauchy sequence* if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a *null sequence* if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the Cauchy sequences with componentwise addition and multiplication form a ring C, and that the null sequences form a maximal ideal N.
Deduce that C/N is a field, with a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

15. Let ϖ be a set of prime numbers. Write $\mathbb{Z}_\varpi$ for the collection of all rationals m/n (in lowest terms) such that the only prime factors of the denominator n are in ϖ.
(i) Show that $\mathbb{Z}_\varpi$ is a subring of the field $\mathbb{Q}$ of rational numbers.
(ii) Show that any subring R of $\mathbb{Q}$ is of the form $\mathbb{Z}_\varpi$ for some set ϖ of primes.
(iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

16. Let F be a field, and let $R = F[X, Y]$ be the polynomial ring in two variables.
(i) Let I be the principal ideal generated by the element $X - Y$ in R. Show that $R/I \cong F[X]$.
(ii) What can you say about R/I when I is the principal ideal generated by $X^2 + Y$?
(iii) What can you say about R/I when I is the principal ideal generated by $X^2 - Y^2$?

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.