IB Groups, Rings and Modules: Example Sheet 1

1. (i) What are the orders of elements of the group S_4? How many elements are there of each order?
(ii) How many subgroups of order 2 are there in S_4? Of order 3? How many cyclic subgroups are there of order 4?
(iii) Find a non-cyclic subgroup V of S_4 of order 4. How many of these are there?
(iv) Find a subgroup D of S_4 of order 8. How many of these are there?

2. (i) Show that A_4 has no subgroups of index 2. Exhibit a subgroup of index 3.
(ii) Show that A_5 has no subgroups of index 2, 3 or 4. Exhibit a subgroup of index 5.
(iii) Show that A_5 is generated by (12)(34) and (135).

3. Calculate the size of the conjugacy class of (123) as an element of S_4, as an element of S_5 and as an element of S_6. Find in each case the centralizer. Hence calculate the size of the conjugacy class of (123) as an element of A_4, as an element of A_5 and as an element of A_6.

4. Suppose that $H, K \lhd G$ with $H \cap K = 1$. Consider the commutator $[h, k] = hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$, and prove that any element of H commutes with any element of K. Hence show that $HK \cong H \times K$.

5. Suppose that G is a non-abelian group of order p^3 where p is prime.
(i) Show that the order of the centre $Z(G)$ is p.
(ii) Show that if $g \notin Z(G)$ then the order of the centralizer $C(g)$ is p^2.
(iii) Hence determine the sizes and numbers of the conjugacy classes.

6. (i) In question 1 we found the number of Sylow 2-subgroups and Sylow 3-subgroups of S_4. Check that your answer is consistent with Sylow's theorems. (Note that if you did not then quite complete proofs for subgroups of order 8, you can do so now.) Identify the normalizers of the Sylow 2-subgroups and Sylow 3-subgroups.
(ii) For $p = 2$, 3, 5 find a Sylow p-subgroup of A_5 and find the normalizer of the subgroup.

7. Show that there is no simple group of order 441. Show that there is no simple group of order 351.
How about orders 300 and 320?

8. Let p, q and r be primes (not necessarily distinct). Show that no group of order pq is simple. Show that no group of order pq^2 is simple. Show that no group of order pqr is simple.

9. (i) Show that any group of order 15 is cyclic.
(ii) Show that any group of order 30 has a normal cyclic subgroup of order 15.

10. Let N and H be groups, and suppose that there is a homomorphism ϕ from H to $Aut(N)$. Show that we can define a group operation on $N \times H$ by
$$(n_1, h_1).(n_2, h_2) = (n_1.n_2^{\phi(h_1)}, h_1.h_2),$$
where we write $n^{\phi(h)}$ for the image of n under $\phi(h)$. Show that the resulting group G has (copies of) N and H as subgroups, that N is normal in G, that $G = NH$ and $N \cap H = 1$.
(We say that G is a semidirect product of N by H.)
Find an element of $Aut(C_7)$ of order 3 and construct a non-abelian group of order 21 as a semidirect product of C_7 by C_3.

Additional Questions

11. Let G be a group of even order with a cyclic Sylow 2-subgroup. By considering the regular action of G, show that G has a normal subgroup of index 2.
[If x is a generator of a Sylow 2-subgroup, show that x is an odd permutation by working out its cycle structure.]

12. Let p be a prime. How many elements of order p are there in S_p, the symmetric group of order p? What are their centralizers? How many Sylow p-subgroups are there? What are the orders of their normalizers? If q is a prime dividing $p-1$, deduce that there exists a non-abelian group of order pq.

13. (Frattini argument) Let P be a Sylow subgroup of the normal subgroup K of G. Show that any element g of G can be written as $g = nk$ with $n \in N_G(P)$ and $k \in K$, and hence $G = N_G(P)K$.
[Observe that P^g is also a Sylow subgroup of K and hence is conjugate to P in K.]
Deduce that G/K is isomorphic to $N_G(P)/N_K(P)$.

14. Show that no non-abelian simple group has order less than 60.

15. Let G be a simple group of order 60. Show that G is isomorphic to the alternating group A_5, as follows. Show that G has six Sylow 5-subgroups. Deduce that G is isomorphic to a subgroup (also denoted by G) of index 6 of the alternating group A_6. By considering the coset action of A_6 on the set of cosets of G in A_6, show that there is an automorphism of A_6 which takes G to A_5.
(The automorphism of A_6 which you have produced has some remarkable properties - it is *not* induced by conjugation by any element of S_6. Such an automorphism of A_n only exists for $n = 6$.)

16. Let G be a group of order 60 which has more than one Sylow 5-subgroup. Show that G must be simple.

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.