Lent 2014 **GROUPS, RINGS AND MODULES – EXAMPLES 4** IBL

All rings are commutative with 1 unless otherwise stated

1. How many abelian groups of order 108 are there?

2. Let M be a module over an integral domain R. An element x of M is a *torsion* element if $rx = 0$ for some non-zero $r \in R$. Prove that the set T of all torsion elements of R is a submodule of M, and that the quotient M/T is torsion-free (meaning that it has no non-zero torsion elements).

3. Let M be a module over a ring R, and let N be a submodule of M. Show that if M is finitely generated then so is M/N. Show also that if N and M/N are finitely generated then so is M.

4. Is the abelian group $\mathbb{Q}$ torsion-free? Is it free? Is it finitely generated?

5. An abelian group is called *indecomposable* if it cannot be written as the direct sum of two non-trivial subgroups. Which finite abelian groups are indecomposable? Write down an infinite abelian group, other than $\mathbb{Z}$, that is indecomposable.

6. Is $C[0,1]$ Noetherian?

7. Show that the image of a Noetherian ring (under a ring homomorphism) is always Noetherian. Use this to give an example of a Noetherian integral domain that is not a UFD. Is every UFD Noetherian?

8. Find a 2×2 matrix over $\mathbb{Z}[X]$ that is not equivalent to a diagonal matrix.

9. Find the Smith normal form for the 4×4 matrix over $\mathbb{Q}[X]$ that is diagonal with entries $X^2 + 2X$, $X^2 + 3X + 2$, $X^3 + 2X^2$, $X^4 + X^3$. What can you deduce from this question about the ability of the lecturer to typeset matrices?

10. Let G be the abelian group given by generators a, b, c and the relations $6a + 10b = 0$, $6a + 15c = 0$, $10b + 15c = 0$ (this means that G is the free abelian group on generators a, b, c quotiented by the subgroup $\langle 6a + 10b, 6a + 15c, 10b + 15c \rangle$). Determine the structure of G as a direct sum of cyclic groups.

11. Let A be a complex matrix with characteristic polynomial $(X + 1)^6(X - 2)^3$ and minimum polynomial $(X + 1)^3(X - 2)^2$. What are the possible Jordan normal forms for A?

12. Let M be a finitely generated module over a ring R, and let f be an R-homomorphism from M to itself. Does f injective imply f surjective? Does f surjective imply f injective?

13. Let R be a ring. Show that, for $m \neq n$, the R-modules R^m and R^n are not isomorphic. What happens if R is not commutative?

$^+$14. Does there exist an abelian group that can be written as the direct sum of two indecomposable subgroups and also as the direct sum of three indecomposable subgroups?