

1. What are the possible cycle types of elements of S_5 ? For each cycle type, determine how many elements have that cycle type, their order, and whether they are even or odd.
2. For $n \geq 5$, let $\sigma \in S_n$ be the 5-cycle (12345). Find the centraliser of σ in S_n . By considering which members of the centraliser belong to A_n , give an alternative proof of the fact that the conjugacy class of σ in A_n is the same as that in S_n for $n \geq 7$, but is half its size for $n = 5$ and $n = 6$.
3. Show that a group of order 441 cannot be simple. Show that a group of order 351 cannot be simple.
4. For a prime p , how many Sylow p -subgroups are there in S_p ? Check that your answer is consistent with Sylow's theorems. Deduce the size of the normaliser of such a subgroup, and describe the normaliser explicitly for the subgroup generated by $(123 \dots p)$.
5. Show that no group of order pqr (where p , q and r are prime) is simple.
6. Let G be a group of order 1001. Prove that G contains normal subgroups of orders 7, 11 and 13, generated by say g_1, g_2, g_3 respectively. By considering expressions of the form $g_i g_j g_i^{-1} g_j^{-1}$, show that the g_i commute with each other. Deduce that G must be cyclic.
7. Let p and q be primes with q dividing $p - 1$. By considering a suitable subgroup of the group G of all maps from \mathbb{Z}_p to itself of the form $x \mapsto ax + b$, where $a, b \in \mathbb{Z}_p$ with $a \neq 0$, show that there exists a non-abelian group of order pq . [Either use your knowledge of the multiplicative group of \mathbb{Z}_p , or else apply Cauchy's theorem to it.]
8. Let G be the group of rotational symmetries of the dodecahedron. Give two Orbit-Stabiliser proofs that G has order 60: one based on the action of G on the vertices and one based on the action of G on the faces. Without knowledge of what this group is, why is it obvious that the group of all symmetries of the dodecahedron cannot be S_5 ?
9. By using the fact that a normal subgroup must be a union of conjugacy classes, prove directly that A_5 and A_6 are simple. Exhibit a subgroup of A_n of index n , and explain how the simplicity of A_n implies that there cannot be a proper subgroup of A_n of smaller index (for $n \geq 5$).
10. Show that a group of order 320 cannot be simple.
11. Is there an infinite simple group?
12. Let G be a group of odd order, and let H be a subgroup of G of index 5. Prove that H is normal.
13. Let G be a finite group acting transitively (i.e. with only one orbit) on a set X . Show that, for $|X| > 1$, some element of G acts with no fixed points. If $|X|$ is a power of 2, show that some element of G acts as an involution (permutation of order 2) with no fixed points.
- +14. For which natural numbers n is there a unique group of order n ?
15. Find an infinite group, other than \mathbb{Z} , that has a unique non-identity automorphism.
+Is there an uncountable group with this property?