

*All rings are commutative with 1 unless otherwise stated*

1. How many abelian groups of order 108 are there?
2. Let  $M$  be a module over an integral domain  $R$ . An element  $x$  of  $M$  is a *torsion* element if  $rx = 0$  for some non-zero  $r \in R$ . Prove that the set  $T$  of all torsion elements of  $M$  is a submodule of  $M$ , and that the quotient  $M/T$  is torsion-free (meaning that it has no non-zero torsion elements).
3. Let  $M$  be a module over a ring  $R$ , and let  $N$  be a submodule of  $M$ . Show that if  $M$  is finitely generated then so is  $M/N$ . Show also that if  $N$  and  $M/N$  are finitely generated then so is  $M$ .
4. Is the abelian group  $\mathbb{Q}$  torsion-free? Is it free? Is it finitely generated?
5. An abelian group is called *indecomposable* if it cannot be written as the direct sum of two non-trivial subgroups. Which finite abelian groups are indecomposable? Write down an infinite abelian group, other than  $\mathbb{Z}$ , that is indecomposable.
6. Is  $C[0, 1]$  Noetherian?
7. Show that the image of a Noetherian ring (under a ring homomorphism) is always Noetherian. Use this to give an example of a Noetherian integral domain that is not a UFD. Is every UFD Noetherian?
8. Find a  $2 \times 2$  matrix over  $\mathbb{Z}[X]$  that is not equivalent to a diagonal matrix.
9. Find the Smith normal form for the  $4 \times 4$  matrix over  $\mathbb{Q}[X]$  that is diagonal with entries  $X^2 + 2X$ ,  $X^2 + 3X + 2$ ,  $X^3 + 2X^2$ ,  $X^4 + X^3$ . What can you deduce from this question about the ability of the lecturer to typeset matrices?
10. Let  $G$  be the abelian group given by generators  $a, b, c$  and the relations  $6a + 10b = 0$ ,  $6a + 15c = 0$ ,  $10b + 15c = 0$  (this means that  $G$  is the free abelian group on generators  $a, b, c$  quotiented by the subgroup  $\langle 6a + 10b, 6a + 15c, 10b + 15c \rangle$ ). Determine the structure of  $G$  as a direct sum of cyclic groups.
11. Let  $A$  be a complex matrix with characteristic polynomial  $(X + 1)^6(X - 2)^3$  and minimum polynomial  $(X + 1)^3(X - 2)^2$ . What are the possible Jordan normal forms for  $A$ ?
12. Let  $M$  be a finitely generated module over a ring  $R$ , and let  $f$  be an  $R$ -homomorphism from  $M$  to itself. Does  $f$  injective imply  $f$  surjective? Does  $f$  surjective imply  $f$  injective?
- +13. Is the set  $\mathbb{Z}^{\mathbb{N}}$  of all integer sequences (with pointwise addition) a free abelian group?
- +14. Does there exist an abelian group that can be written as the direct sum of two indecomposable subgroups and also as the direct sum of three indecomposable subgroups?