

All rings are commutative with 1 unless otherwise stated

1. Show that $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\omega]$ are Euclidean domains, where $\omega = (1 + \sqrt{-3})/2$. Show also that the usual Euclidean function $\phi(r) = N(r)$ does not make $\mathbb{Z}[\sqrt{-3}]$ into a Euclidean domain. Could there be some other Euclidean function ϕ making $\mathbb{Z}[\sqrt{-3}]$ into a Euclidean domain?
2. Show that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain.
3. Exhibit an element of $\mathbb{Z}[\sqrt{-17}]$ that is a product of two irreducibles and also a product of three irreducibles.
4. Show that if R is an integral domain then a polynomial in $R[X]$ of degree d can have at most d roots. Give a quadratic polynomial in $\mathbb{Z}_8[X]$ that has more than two roots.
5. Determine whether or not the following rings are fields, PIDs, UFDs, integral domains:
 $\mathbb{Z}[X]$, $\mathbb{Z}[X]/(X^2 + 1)$, $\mathbb{Z}_2[X]/(X^2 + 1)$, $\mathbb{Z}_2[X]/(X^2 + X + 1)$, $\mathbb{Z}_3[X]/(X^2 + X + 1)$
6. Determine which of the following polynomials are irreducible in $\mathbb{Q}[X]$:
 $X^4 + 2X + 2$, $X^4 + 18X^2 + 24$, $X^3 - 9$, $X^3 + X^2 + X + 1$, $X^4 + 1$, $X^4 + 4$
7. A complex number α is an *algebraic integer* if it is a root of a monic polynomial $f \in \mathbb{Z}[X]$. Explain why we may assume that f is irreducible. Prove carefully that $\mathbb{Z}[X]/(f)$ is isomorphic to $\mathbb{Z}[\alpha]$, the subring of \mathbb{C} generated by (\mathbb{Z} and) α – in other words, that quotienting the polynomials over \mathbb{Z} by (f) may be viewed as ‘adding a root of f to \mathbb{Z} ’.
8. Give two elements of $\mathbb{Z}[\sqrt{-5}]$ that do not have an HCF.
9. Explain why, in a PID, the HCF of two elements a and b may always be written as a linear combination of a and b (i.e. as $xa + yb$, some x, y), and give an example in $\mathbb{Z}[X]$ of two elements whose HCF cannot be written in this way. In a Euclidean domain, what would the ‘Euclidean algorithm’ for calculating HCFs be? Find the HCF of $11 + 7i$ and $18 - i$ in $\mathbb{Z}[i]$.
10. By considering factorisations in $\mathbb{Z}[\sqrt{-2}]$, show that the equation $x^2 + 2 = y^3$ has no solutions in integers except for $x = \pm 5$, $y = 3$.
11. Exhibit an integral domain R and a (non-zero, non-unit) element of R that is not a product of irreducibles.
12. If a UFD has at least one irreducible, must it have infinitely many (pairwise non-associate) irreducibles?
13. Let R be a ring on ground-set \mathbb{Z} whose multiplication is the same as the usual multiplication on \mathbb{Z} . Must its addition be the same as the usual addition on \mathbb{Z} ?
- +14. Let R be a Euclidean domain in which the quotient and remainder are always unique (in other words, for any a and b with $b \neq 0$ there exist unique q and r with $a = bq + r$ and $\phi(r) < \phi(b)$ or $r = 0$). Does it follow that the ring R is either a field or a polynomial ring $F[X]$ for some field F ?