Lent 2013 **GROUPS, RINGS AND MODULES – EXAMPLES 2** IBL

All rings are commutative with 1 unless otherwise stated

1. Let $\omega = (1+\sqrt{-3})/2$, and let $R = \{a + b\omega : \ a, b \in \mathbb{Z}\}$. Show that R is a ring. What are the units of R?

2. Show that a (non-trivial) ring R is a field if and only if the only ideals of R are $\{0\}$ and R.

3. An element r of a ring R is *nilpotent* if $r^n = 0$ for some n. What are the nilpotent elements of $\mathbb{Z}_6$? Of $\mathbb{Z}_8$? Of $\mathbb{Z}_{24}$? Show that if r is nilpotent then $1 + r$ is a unit. Show also that the nilpotent elements form an ideal.

4. Let r be an element of a ring R. Show that, in $R[X]$, the polynomial $1 + rX$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

5. Show that if p and q are coprime then the ring $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is isomorphic to $\mathbb{Z}_{pq}$.

6. Let A and B be elements of the power-set ring $\mathbb{P}(S)$, for some set S. What is the principal ideal (A)? What is the ideal (A, B)?

7. Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be ideals in a ring R. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.

8. Let R be the ring $C[0, 1]$ of continuous real-valued functions on $[0, 1]$, and let $I = \left\{f \in R : \ f(x) = 0 \text{ for } 0 \leq x \leq \frac{1}{2}\right\}$. Show that I is an ideal. What is R/I?

9. Show that, for each $0 \leq x \leq 1$, the set of functions vanishing at x is a maximal ideal of $C[0, 1]$. Prove that all maximal ideals of $C[0, 1]$ are of this form.

10. An element r of a ring R is *idempotent* if $r^2 = r$. What are the idempotent elements of $\mathbb{Z}_6$? Of $\mathbb{Z}_8$? Of $\mathbb{Z}_{24}$? Show that if r is idempotent then so is $1 - r$. Show also that if r is idempotent then (r) is naturally a ring, and that R is isomorphic to $(r) \oplus (1 - r)$.

11. A ring R is *Boolean* if every element of R is idempotent. For each n, exhibit a Boolean ring of order 2^n. Prove that every finite Boolean ring is isomorphic to a power-set ring $\mathbb{P}(S)$, for some set S. Give an example to show that this need not remain true for infinite Boolean rings.

12. Show that if an element of a (not necessarily commutative) ring has two right-inverses then it has infinitely many.

13. For each n, give (with proof) an ideal of $\mathbb{Z}[X]$ that is generated by n elements but not by $n - 1$ elements.

14. Is every abelian group the additive group of some ring?