Lent 2013 **GROUPS, RINGS AND MODULES – EXAMPLES 1** IBL

1. What are the possible cycle types of elements of S_5? For each cycle type, determine how many elements have that cycle type, their order, and whether they are even or odd.

2. For $n \geq 5$, let $\sigma \in S_n$ be the 5-cycle (12345). Find the centraliser of σ in S_n. By considering which members of the centraliser belong to A_n, give an alternative proof of the fact that the conjugacy class of σ in A_n is the same as that in S_n for $n \geq 7$, but is half its size for $n = 5$ and $n = 6$.

3. Show that a group of order 441 cannot be simple. Show that a group of order 351 cannot be simple.

4. For a prime p, how many Sylow p-subgroups are there in S_p? Check that your answer is consistent with Sylow's theorems. Deduce the size of the normaliser of such a subgroup, and describe the normaliser explicitly for the subgroup generated by $(123\ldots p)$.

5. Show that no group of order pqr (where p, q and r are prime) is simple.

6. Let G be a group of order 1001. Prove that G contains normal subgroups of orders 7, 11 and 13, generated by say g_1, g_2, g_3 respectively. By considering expressions of the form $g_i g_j g_i^{-1} g_j^{-1}$, show that the g_i commute with each other. Deduce that G must be cyclic.

7. Let p and q be primes with q dividing $p - 1$. By considering a suitable subgroup of the group G of all maps from $\mathbb{Z}_p$ to itself of the form $x \mapsto ax + b$, where $a, b \in \mathbb{Z}_p$ with $a \neq 0$, show that there exists a non-abelian group of order pq. [Either use your knowledge of the multiplicative group of $\mathbb{Z}_p$, or else apply Cauchy's theorem to it.]

8. Let G be the group of rotational symmetries of the dodecahedron. Give two Orbit-Stabiliser proofs that G has order 60: one based on the action of G on the vertices and one based on the action of G on the faces. Without knowledge of what this group is, why is it obvious that the group of all symmetries of the dodecahedron cannot be S_5?

9. By using the fact that a normal subgroup must be a union of conjugacy classes, prove directly that A_5 and A_6 are simple. Exhibit a subgroup of A_n of index n, and explain how the simplicity of A_n implies that there cannot be a proper subgroup of A_n of smaller index (for $n \geq 5$).

10. Show that a group of order 320 cannot be simple.

11. Let G be a group of odd order, and let H be a subgroup of G of index 5. Prove that H is normal.

12. Is there an infinite simple group?

$^+$13. For which natural numbers n is there a unique group of order n?

14. Which finite groups have the property that all non-identity elements are conjugate? $^+$Is there an infinite group with this property?