Lent Term 2011 R. Camina

# IB Groups, Rings and Modules: Example Sheet 2

All rings in this course are commutative with a multiplicative identity.

1. Let $\omega = \frac{1}{2}(1 + \sqrt{-3})$, let $R = \{a + b\omega : \ a, b \in \mathbb{Z}\}$, and let $F = \{a + b\omega : \ a, b \in \mathbb{Q}\}$. Show that $R$ is a subring of $\mathbb{C}$, and that $F$ is a subfield of $\mathbb{C}$. What are the units of $R$?

2. *An element $r$ of a ring $R$ is nilpotent if $r^n = 0$ for some $n$.*
(i) What are the nilpotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
(ii) Show that if $r$ is nilpotent then $r$ is not a unit, but $1 + r$ and $1 - r$ are units.
(iii) Show that the nilpotent elements form an ideal $N$ in $R$. What are the nilpotent elements in the quotient ring $R/N$?

3. Let $r$ be an element of a ring $R$. Show that, in the polynomial ring $R[X]$, the polynomial $1 + rX$ is a unit if and only if $r$ is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

4. Show that if $I$ and $J$ are ideals in the ring $R$, then so is $I \cap J$, and the quotient $R/(I \cap J)$ is isomorphic to a subring of the product $R/I \times R/J$.

5. (i) *A proper ideal $P$ of the ring $R$ is prime if $rs \in P \Rightarrow r \in P$ or $s \in P$, for all $r, s \in R$.*
Let $I$ be an ideal of the ring $R$ and $P_1, \ldots, P_n$ be prime ideals of $R$. Show that if $I \subset \bigcup_{i=1}^{n} P_i$ , then $I \subset P_i$ for some $i$.
(ii) *A proper ideal $M$ of the ring $R$ is maximal if no proper ideal strictly contains it (i.e. $M \subset I \subset R \Rightarrow I = M$ or $I = R$).*
Show that $(2, X)$ is maximal in $\mathbb{Z}[X]$ but that $(2, X^2 + 1)$ is not.
(iii) Show that a maximal ideal is a prime ideal.

6. Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be ideals in a ring $R$. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each $I_n$ is proper, explain why $I$ must be proper. If each $I_n$ is prime, show that $I$ must be prime.

7. Let $R$ be an integral domain and $F$ be its field of fractions. Suppose that $\phi : R \to K$ is an injective ring homomorphism from $R$ to a field $K$. Show that $\phi$ extends to an injective homomorphism $\Phi : F \to K$ from $F$ to $K$. What happens if we do not assume that $\phi$ is injective?

8. Let $R$ be any ring. Show that the ring $R[X]$ is a principal ideal domain if and only if $R$ is a field.

9. Show that a finite integral domain is a field.

10. *An element $r$ of a ring $R$ is idempotent if $r^2 = r$.*
(i) What are the idempotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
(ii) Show that if $r$ is idempotent then so is $r' = 1 - r$, and $rr' = 0$. Show also that the ideal $(r)$ is naturally a ring, and that $R$ is isomorphic to $(r) \times (r')$.

11. Show that the set $P(S)$ of all subsets of a given set $S$ is a ring with respect to the operations of symmetric difference and intersection. Describe the principal ideals in this ring. Show that the ideal $(A, B)$ generated by elements $A, B$ is in fact principal. Are there any non-principal ideals?

12. By writing out the addition and multiplication tables, construct a field of order 4. Can you construct a field of order 6?

## Additional Questions

13. Is every abelian group the additive group of some ring?

14. Let $P$ be a prime ideal of $R$. Prove that $P[X]$ is a prime ideal of $R[X]$. If $M$ is a maximal ideal of $R$, does it follow that $M[X]$ is a maximal ideal of $R[X]$?

15. A sequence $\{a_n\}$ of rational numbers is a *Cauchy sequence* if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a *null sequence* if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the Cauchy sequences with componentwise addition and multiplication form a ring $C$, and that the null sequences form a maximal ideal $N$.
Deduce that $C/N$ is a field, with a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

16. Let $\varpi$ be a set of prime numbers. Write $\mathbb{Z}_\varpi$ for the collection of all rationals $m/n$ (in lowest terms) such that the only prime factors of the denominator $n$ are in $\varpi$.
(i) Show that $\mathbb{Z}_\varpi$ is a subring of the field $\mathbb{Q}$ of rational numbers.
(ii) Show that any subring $R$ of $\mathbb{Q}$ is of the form $\mathbb{Z}_\varpi$ for some set $\varpi$ of primes.
(iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

17. Let $F$ be a field, and let $R = F[X, Y]$ be the polynomial ring in two variables.
(i) Let $I$ be the principal ideal generated by the element $X - Y$ in $R$. Show that $R/I \cong F[X]$.
(ii) What can you say about $R/I$ when $I$ is the principal ideal generated by $X^2 + Y$?
(iii) [Harder] What can you say about $R/I$ when $I$ is the principal ideal generated by $X^2 - Y^2$?

+18. Does every ring have a maximal ideal?

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.