IB Groups, Rings and Modules: Example Sheet 2

All rings in this course are commutative with a multiplicative identity.

1. Let $\omega = (1+\sqrt{-3})/2$, and let R be the set $\{a + b\omega : \ a, b \in \mathbb{Z}\}$. Show that R is a subring of the ring $\mathbb{C}$. What are the units of R?

2. An element r of a ring R is *nilpotent* if $r^n = 0$ for some n.
(i) What are the nilpotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$?
(ii) Show that if r is nilpotent then r is not a unit, but $1 + r$ and $1 - r$ are units.
(iii) Show that the nilpotent elements form an ideal in R.

3. Let r be an element of a ring R. Show that, in the polynomial ring $R[X]$, the polynomial $1+rX$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

4. Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be ideals in a ring R. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.

5. (i) Show that if I and J are ideals in the ring R, then so is $I \cap J$, and the quotient $R/(I \cap J)$ is isomorphic to a subring of the product $R/I \times R/J$.
(ii) Show that if p and q are coprime integers, then $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ is isomorphic to $\mathbb{Z}/pq\mathbb{Z}$.

6. Let R be the ring $C[0, 1]$ of continuous real-valued functions on $[0, 1]$, and let
$I = \{f \in R : \ f(x) = 0 \text{ for } 0 \le x \le \frac{1}{2}\}$. Show that I is an ideal. What is R/I?

7. Let R be an integral domain and F be its field of fractions. Suppose that $\phi : R \to K$ is an injective ring homomorphism from R to a field K. Show that ϕ extends to an injective homomorphism $\Phi : F \to K$ from F to K. What happens if we do not assume that ϕ is injective?

8. Let R be any ring. Show that the ring $R[X]$ is a principal ideal domain if and only if R is a field.

9. An element r of a ring R is *idempotent* if $r^2 = r$.
(i) What are the idempotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$?
(ii) Show that if r is idempotent then so is $r' = 1 - r$, and $r \cdot r' = 0$.
(iii) Show also that if r is idempotent then the ideal (r) is naturally a ring, and that R is isomorphic to $(r) \times (r')$.

10. (i) Show that the set $P(S)$ of all subsets of a given set S is a ring with respect to the operations of symmetric difference and intersection - the power-set ring. Note that in this ring $A^2 = A$
for all elements A. Describe the principal ideals in this ring. Describe the ideal (A, B) generated by elements A, B.
(ii) The ring R is *Boolean* if every element of R is idempotent. Prove that every finite Boolean ring is isomorphic to a power-set ring $P(S)$ for some set S.
Give an example to show that this need not remain true for infinite Boolean rings.

Additional Questions

11. A sequence $\{a_n\}$ of rational numbers is a *Cauchy sequence* if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a *null sequence* if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the Cauchy sequences with componentwise addition and multiplication form a ring C, and that the null sequences form a maximal ideal N.
Deduce that C/N is a field, with a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

12. Let ϖ be a set of prime numbers. Write $\mathbb{Z}_\varpi$ for the collection of all rationals m/n (in lowest terms) such that the only prime factors of the denominator n are in ϖ.
(i) Show that $\mathbb{Z}_\varpi$ is a subring of the field $\mathbb{Q}$ of rational numbers.
(ii) Show that any subring R of $\mathbb{Q}$ is of the form $\mathbb{Z}_\varpi$ for some set ϖ of primes.
(iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

13. Let F be a field, and let $R = F[X, Y]$ be the polynomial ring in two variables.
(i) Let I be the principal ideal generated by the element $X - Y$ in R. Show that $R/I \cong F[X]$.
(ii) What can you say about R/I when I is the principal ideal generated by $X^2 + Y$?
(iii) [Harder] What can you say about R/I when I is the principal ideal generated by $X^2 - Y^2$?

Comments and corrections should be sent to `rdc26@dpmms.cam.ac.uk`.