IB Groups, Rings and Modules: Example Sheet 4

All rings in this course are commutative with a multiplicative identity.

1. Let M be a module over an integral domain R. An element m is a *torsion* element if $rm = 0$ for some non-zero $r \in R$. Show that the set of T of all torsion elements in M is a submodule of M - the *torsion submodule*. Show further that the quotient M/T is *torsion-free*, that is, the only torsion element is the zero element.

2. Show that if N is a submodule of the module M, and if both N and M/N are finitely generated, then M is finitely generated.

3. We say that an R-module satisfies condition (N) on submodules if any submodule is finitely generated. Show that this condition is equivalent to condition (ACC): every increasing chain of submodules terminates.

4. (i) Prove that $\mathbb{Q}$ is not finitely generated as a module over $\mathbb{Z}$.
   (ii) Prove that $\mathbb{R}$ is not finitely generated as a module over $\mathbb{Q}$.

5. Use elementary operations to bring the integer matrix $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$ to Smith normal form D.
   Check your result using minors. Write down invertible matrices P, Q for which $D = QAP$.

6. Work out the invariant factors of the matrices over $\mathbb{R}[X]$:
$$\begin{pmatrix} 2X-1 & X & X-1 & 1 \\ X & 0 & 1 & 0 \\ 0 & 1 & X & X \\ 1 & X^2 & 0 & 2X-2 \end{pmatrix} \text{ and } \begin{pmatrix} X^2+2X & 0 & 0 & 0 \\ 0 & (X+2)(X+1) & 0 & 0 \\ 0 & 0 & X^3+2X^2 & 0 \\ 0 & 0 & 0 & X^4+X^3 \end{pmatrix}.$$

7. Let A be an abelian group generated by a and b subject to the relation $6a + 9b = 0$. Determine the structure of A as a direct sum of cyclic groups.

8. How many abelian groups are there of order 6? Of order 60? Of order 6000?

9. Write $f(n)$ for the number of distinct abelian groups of order n.
   (i) Show that if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ with the p_i distinct primes and $a_i \in \mathbb{N}$ then $f(n) = f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_k^{a_k})$.
   (ii) Show that $f(p^a)$ equals the number $p(a)$ of partitions of a, that is, $p(a)$ is the number of ways of writing a as a sum of positive integers, where the order of summands is unimportant. (For example, $p(5) = 7$, since $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$.)

10. Let A be a complex matrix with characteristic polynomial $(X+1)^6(X-2)^3$ and minimal polynomial $(X+1)^3(X-2)^2$. Write down the possible Jordan normal forms for A.

11. A real $n \times n$ matrix A satisfies the equation $A^2 + I = 0$. Show that n is even and A is similar to a block matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ with each block an $m \times m$ matrix (where $n = 2m$).

Additional Questions

12. Let R be a Noetherian ring and M be a finitely generated R-module. Show that all submodules of M are finitely generated.

13. Show that a complex number α is an algebraic integer if and only if the additive group of the ring $\mathbb{Z}[\alpha]$ is finitely generated (i.e. $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module). Furthermore if α and β are algebraic integers show that the subring $\mathbb{Z}[\alpha, \beta]$ of $\mathbb{C}$ generated by α and β also has a finitely generated additive group and deduce that $\alpha - \beta$ and $\alpha\beta$ are algebraic integers. Show that the algebraic integers form a subring of $\mathbb{C}$.

14. What is the rational canonical form of a matrix?

    Show that the group $GL_2(\mathbb{F}_2)$ of non-singular 2×2 matrices over the field $\mathbb{F}_2$ of 2 elements has three conjugacy classes of elements.

    Show that the group $GL_3(\mathbb{F}_2)$ of non-singular 3×3 matrices over the field $\mathbb{F}_2$ has six conjugacy classes of elements, corresponding to minimal polynomials $X+1, (X+1)^2, (X+1)^3, X^3+1, X^3+X^2+1, X^3+X+1$, one each of elements of orders $1, 2, 3$ and 4, and two of elements of order 7.

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