

## IB Groups, Rings and Modules: Example Sheet 1

1. (i) What are the orders of elements of the group  $S_4$ ? How many elements are there of each order?  
 (ii) How many subgroups of order 2 are there in  $S_4$ ? Of order 3? How many cyclic subgroups are there of order 4?  
 (iii) Find a non-cyclic subgroup  $V$  of  $S_4$  of order 4. How many of these are there?  
 (iv) Find a subgroup  $D$  of  $S_4$  of order 8. How many of these are there?
2. You should have identified three subgroups of order 8 in  $S_4$ .  
 (i) The group of rotations of the cube is isomorphic to  $S_4$ . Identify a geometrical feature such that the rotations preserving it form a subgroup of order 8.  
 (ii) The group of symmetries of the tetrahedron is isomorphic to  $S_4$ . Identify a geometrical feature such that the symmetries preserving it form a subgroup of order 8.  
 [Hint. In each case there should be three of these features. Why?]
3. (i) Show that  $A_4$  has no subgroups of index 2. Exhibit a subgroup of index 3.  
 (ii) Show that  $A_5$  has no subgroups of index 2, 3 or 4. Exhibit a subgroup of index 5.  
 (iii) Show that  $A_5$  is generated by (12)(34) and (135). (Multiply the two elements to show that the subgroup they generate has order 30 or 60.)
4. Calculate the size of the conjugacy class of (123) as an element of  $S_4$ , as an element of  $S_5$  and as an element of  $S_6$ . Find in each case the centralizer. Hence calculate the size of the conjugacy class of (123) as an element of  $A_4$ , as an element of  $A_5$  and as an element of  $A_6$ .
5. Suppose that  $H, K \triangleleft G$  with  $H \cap K = 1$ . Consider the commutator  $[h, k] = hkh^{-1}k^{-1}$  with  $h \in H$  and  $k \in K$ , and prove that any element of  $H$  commutes with any element of  $K$ . Hence show that  $HK \cong H \times K$ .
6. Suppose that  $G$  is a non-abelian group of order  $p^3$  where  $p$  is prime.  
 (i) Show that the order of the centre  $Z(G)$  is  $p$ , and that  $G/Z(G) \cong C_p \times C_p$ .  
 (ii) Show that if  $g \notin Z(G)$  then the order of the centralizer  $C(g)$  is  $p^2$ .  
 (iii) Hence determine the sizes and numbers of the conjugacy classes.
7. (i) In question 1 we found the number of Sylow 2-subgroups and Sylow 3-subgroups of  $S_4$ . Check that your answer is consistent with Sylow's theorems. (Note that if you did not then quite complete proofs for subgroups of order 8, you can do so now.) Identify the normalizers of the Sylow 2-subgroups and Sylow 3-subgroups.  
 (ii) For  $p = 2, 3, 5$  find a Sylow  $p$ -subgroup of  $A_5$  and find the normalizer of the subgroup.
8. Let  $p, q$  and  $r$  be primes. Show that no group of order  $pq$  is simple. Show that no group of order  $pq^2$  is simple. Show that no group of order  $pqr$  is simple.
9. (i) Show that any group of order 15 is cyclic.  
 (ii) Show that any group of order 30 has a normal cyclic subgroup of order 15.
10. Let  $N$  and  $H$  be groups, and suppose that there is a homomorphism  $\phi$  from  $H$  to  $Aut(N)$ . Show that we can define a group operation on the set of pairs  $(n, h)$ , for  $n \in N$  and  $h \in H$ , by

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot n_2^{\phi(h_1)}, h_1 \cdot h_2),$$

where we write  $n^{\phi(h)}$  for the image of  $n$  under  $\phi(h)$ . Show that the resulting group  $G$  has (copies of)  $N$  and  $H$  as subgroups, that  $N$  is normal in  $G$ , that  $G = NH$  and  $N \cap H = 1$ .

(We say that  $G$  is a semidirect product of  $N$  by  $H$ .) Find an element of  $Aut(C_7)$  of order 3 and construct a non-abelian group of order 21 as a semidirect product of  $C_7$  by  $C_3$ .

### Additional Questions

11. (i) Let  $G$  be a group acting faithfully on a set  $\Omega$  with orbits  $\Omega_1, \Omega_2, \dots, \Omega_r$ . Show that  $G$  is isomorphic to a subgroup of the direct product  $Sym(\Omega_1) \times \dots \times Sym(\Omega_r)$ .  
(ii) Let  $G$  be a subgroup of  $A_5$  of order 12. Show that there is one orbit of size 1 and one of size 4 under the canonical action of  $G$  on  $\{1, \dots, 5\}$ . Deduce that  $G$  is isomorphic to  $A_4$  and that there are 5 such subgroups  $G$  in  $A_5$ .
12. Show that no non-abelian simple group has order less than 60.
13. Let  $G$  be a group of even order with a cyclic Sylow 2-subgroup. By considering the regular action of  $G$ , show that  $G$  has a normal subgroup of index 2.  
[If  $x$  is a generator of a Sylow 2-subgroup, show that  $x$  is an odd permutation by working out its cycle structure.]
14. Let  $p$  be a prime. How many elements of order  $p$  are there in  $S_p$ , the symmetric group of order  $p$ ? What are their centralizers? How many Sylow  $p$ -subgroups are there? What are the orders of their normalizers? If  $q$  is a prime dividing  $p - 1$ , deduce that there exists a non-abelian group of order  $pq$ .
15. (Frattini argument) Let  $P$  be a Sylow subgroup of the normal subgroup  $K$  of  $G$ . Show that any element  $g$  of  $G$  can be written as  $g = nk$  with  $n \in N_G(P)$  and  $k \in K$ , and hence  $G = N_G(P)K$ .  
[Observe that  $P^g$  is also a Sylow subgroup of  $K$  and hence is conjugate to  $P$  in  $K$ .]  
Deduce that  $G/K$  is isomorphic to  $N_G(P)/N_K(P)$ .
16. Let  $G$  be a simple group of order 60. Show that  $G$  is isomorphic to the alternating group  $A_5$ , as follows. Show that  $G$  has six Sylow 5-subgroups. Deduce that  $G$  is isomorphic to a subgroup (also denoted by  $G$ ) of index 6 of the alternating group  $A_6$ . By considering the coset action of  $A_6$  on the set of cosets of  $G$  in  $A_6$ , show that there is an automorphism of  $A_6$  which takes  $G$  to  $A_5$ .  
(The automorphism of  $A_6$  which you have produced has some remarkable properties - it is *not* induced by conjugation by any element of  $S_6$ . Such an automorphism of  $A_n$  only exists for  $n = 6$ .)
17. Let  $G$  be a group of order 60 which has more than one Sylow 5-subgroup. Show that  $G$  must be simple.

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