Lent Term 2005 C.J.B. Brookes

IB Groups, Rings and Modules: Example Sheet 1

This sheet covers most of the first part of the course which is on Group Theory. Completing the first ten questions should give you a good grasp of the material. The additional questions are not intended to be much more difficult, but to provide further practice.

1. (i) What are the orders of elements of the group S_4? How many elements are there of each order?
(ii) How many subgroups of order 2 are there in S_4? Of order 3? How many cyclic subgroups are there of order 4?
(iii) Find a non-cyclic subgroup V of S_4 of order 4. How many of these are there?
(iv) Find a subgroup D of S_4 of order 8. How many of these are there?

2. Suppose that $f : X \to Y$ is a map of sets. Let R be the equivalence relation on X defined by aRb if and only if $f(a) = f(b)$. Show that there is a bijection $\bar{f} : X/R \to \mathrm{Im} f$ such that f factorizes as the composite
$$X \to X/R \to \mathrm{Im} f \to Y.$$
[You should compare this with the First Isomorphism Theorem.]

3. (i) Show that A_4 has no subgroups of index 2. Exhibit a subgroup of index 3.
(ii) Show that A_5 has no subgroups of index 2, 3 or 4. Exhibit a subgroup of index 5.
(iii) Show that A_5 is generated by (12)(34) and (135). (Multiply the two elements to show that the subgroup they generate has order 30 or 60.)

4. Suppose that $H, K \lhd G$ with $H \cap K = 1$. Consider the commutator $[h, k] = hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$, and prove that any element of H commutes with any element of K. Hence show that $HK \cong H \times K$.

5. Let N and H be groups, and suppose that there is a homomorphism ϕ from H to $Aut(N)$. Show that we can define a group operation on the set $N \times H$ by
$$(n_1, h_1).(n_2, h_2) = (n_1.n_2^{\phi(h_1)}, h_1.h_2),$$
where we write $n^{\phi(h)}$ for the image of n under $\phi(h)$. Show that the resulting group G has (copies of) N and H as subgroups, that N is normal in G, that $G = NH$ and $N \cap H = 1$.
(We say that G is a semidirect product of N by H.) Show that the dihedral group D_{2n} is isomorphic to a semidirect product of a group of order n by a group of order 2.

6. Suppose that G is a non-abelian group of order p^3 where p is prime.
(i) Show that the order of the centre $Z(G)$ is p, and that $G/Z(G) \cong C_p \times C_p$.
(ii) Show that if $g \notin Z(G)$ then the order of the centralizer $C(g)$ is p^2.
(iii) Hence determine the sizes and numbers of the conjugacy classes.

7. (i) In question 1 we found the number of Sylow 2-subgroups and Sylow 3-subgroups of S_4. Check that your answer is consistent with Sylow's theorems. (Note that if you did not then quite complete proofs for subgroups of order 8, you can do so now.) Identify the normalizers of the Sylow 2-subgroups and Sylow 3-subgroups.
(ii) For $p = 2$, 3, 5 find a Sylow p-subgroup of A_5 and find the normalizer of the subgroup.

8. Let p, q and r be primes. Show that no group of order pq is simple. Show that no group of order pq^2 is simple. Show that no group of order pqr is simple.

9. Show that no non-abelian simple group has order less than 60.

10. (i) Show that any group of order 15 is cyclic.
(ii)Show that any group of order 30 has a normal cyclic subgroup of order 15.

Additional Questions

11. You should have identified three Sylow 2-subgroups of S_4.
(i) The group of rotations of the cube is isomorphic to S_4. Identify a geometrical feature such that the rotations preserving it form a subgroup of order 8.
(ii) The group of symmetries of the tetrahedron is isomorphic to S_4. Identify a geometrical feature such that the symmetries preserving it form a subgroup of order 8.
[Hint. In each case there should be three of these features. Why?]

12. Let G be a group of even order with a cyclic Sylow 2-subgroup. By considering the regular action of G, show that G has a normal subgroup of index 2.
[If x is a generator of a Sylow 2-subgroup, show that x is an odd permutation by working out its cycle structure.]

13. Let p be a prime. How many elements of order p are there in S_p, the symmetric group of order p? What are their centralizers? How many Sylow p-subgroups are there? What are the orders of their normalizers? If q is a prime dividing $p-1$, deduce that there exists a non-abelian group of order pq.

14. (Frattini argument) Let P be a Sylow subgroups of the normal subgroup K of G. Show that any element g of G can be written as $g = nk$ with $n \in N_G(P)$ and $k \in K$, and hence $G = N_G(P)K$.
[Observe that P^g is also a Sylow subgroup of K and hence is conjugate to P in K.]
Deduce that G/K is isomorphic to $N_G(P)/N_K(P)$.

15. Let G be a simple group of order 60. Show that G is isomorphic to the alternating group A_5, as follows. Show that G has six Sylow 5-subgroups. Deduce that G is isomorphic to a subgroup (also denoted by G) of index 6 of the alternating group A_6. By considering the coset action of A_6 on the set of cosets of G in A_6, show that there is an automorphism of A_6 which takes G to A_5.
(The automorphism of A_6 which you have produced has some remarkable properties - it is *not* induced by conjugation by any element of S_6. Such an automorphism of A_n only exists for $n = 6$.)

16. Let G be a group of order 60 which has more than one Sylow 5-subgroup. Show that G must be simple.

Comments and corrections should be sent to `brookes@dpmms.cam.ac.uk`.