Geometry IB – Lent 2025 – Sheet 4: *Hyperbolic surfaces and Gauss-Bonnet*

1. Show that a non-identity Möbius transformation T has exactly one or two fixed points in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Show that if T corresponds, under stereographic projection, to a rotation of S^2, then it has two fixed points z_i which satisfy $z_2 = -1/\bar{z}_1$. If $T \in$ Möb has two fixed points z_i and $z_2 = -1/\bar{z}_1$, prove that either T corresponds to a rotation, or one of the two fixed points (say z_1) is attractive, i.e. $T^n(z) \to z_1$ for all $z \neq z_2$ as $n \to \infty$.

2. (i) Show that inversion in the circle $\{|z-a| = r\}$ is given by $z \mapsto a + \frac{r^2}{\bar{z}-\bar{a}}$.

   (ii) Show that every Möbius map is a composition of inversions.

3. Show from first principles that a vertical line segment is length-minimizing and hence defines a geodesic in the hyperbolic upper half-plane, i.e. the upper half-plane with the abstract metric $\frac{dx^2+dy^2}{y^2}$.

4. Let z_1, z_2 be distinct points in the upper half plane $\mathfrak{h}$. Suppose that the hyperbolic line through z_1 and z_2 meets the real axis at points w_1 and w_2, where z_1 lies on the hyperbolic line segment $w_1 z_2$ (and where one w_i may be ∞). Show that the hyperbolic distance $d_{hyp}(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2 taken in an appropriate order.

5. (a) Let $P \in S^2 \subset \mathbb{R}^3$ be a point on the round sphere. The spherical circle with centre P and radius ρ is the set $\{w \in S^2 \,|\, d_{sph}(w, P) = \rho\}$, where d_{sph} is the spherical metric (induced by the first fundamental form of the embedding). Prove that a spherical circle of radius ρ is a Euclidean circle. Prove that its circumference is $2\pi \sin(\rho)$ and that it bounds a disc on S^2 of area $2\pi(1-\cos(\rho))$.

   (b) Let $C \subset \mathfrak{h}$ be a hyperbolic circle with centre $p \in \mathfrak{h}$ and radius ρ, i.e. the locus $\{w \in \mathfrak{h} \,|\, d_{hyp}(w, p) = \rho\}$ for some $\rho > 0$. Show that C is a Euclidean circle. If $p = ic$ for $c \in \mathbb{R}_{>0}$, find the centre and radius of C as a Euclidean circle. Show the hyperbolic circumference of C is $2\pi \sinh(\rho)$, and the hyperbolic area of the disc it bounds is $2\pi(\cosh(\rho)-1)$. Deduce that no hyperbolic triangle contains a hyperbolic circle of radius $> \cosh^{-1}(3/2)$.

   (c) Show that there is some $\delta > 0$ such that, in any hyperbolic triangle, the union of the δ-neighbourhoods of two of the sides completely contains the 3rd side. (Does such a δ exist for triangles in the Euclidean plane?)

6. Fix a hyperbolic triangle $\Delta \subset \mathbb{H}^2$ with interior angles A, B, C and side lengths (in the hyperbolic metric) a, b, c, where a is the side opposite the vertex with angle A, etc.

   (a) Suppose that C is a right-angle. By applying the 'hyperbolic cosine' formula in two different ways, prove that

   $$\sin(A)\sinh(c) = \sinh(a).$$

   (b) Deduce that for a general hyperbolic triangle, one has

   $$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$$

7. (a) Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the common perpendicular is unique. Show that, up to isometry, for $t > 0$ there is a unique configuration of ultraparallel lines for which the segment of the common perpendicular between the lines has length t.

   (b) Let l_1, l_2 be ultraparallel hyperbolic lines, and let r_{l_i} denote the hyperbolic isometry given by reflection in l_i. Prove that $r_{l_1} \circ r_{l_2}$ has infinite order.

8. (a) Consider the 'ideal' hyperbolic square with vertices at $0, 1, \infty, -1$ in the upper half-plane model. By gluing the edges of the square by isometries, or otherwise, prove that there is a *complete* hyperbolic metric on the smooth surface $S^2 \backslash \{p, q, r\}$ given by the complement of 3 distinct points in the sphere.

   (b) Construct a non-orientable compact hyperbolic surface.

The questions $E1 - E3$ are 'extras'; they may be harder or go beyond the core course material or both, but are included for the interested. They are recommended only for those already comfortable with the preceding questions (and supervisors might or might not feel inclined to talk about them).

E1 (a) Let A, B be disjoint circles in $\mathbb{C}$. Show that there is a Möbius transformation which takes A and B to two concentric circles.

(b) A collection of circles $X_i \subset \mathbb{C}$, $0 \leq i \leq n-1$, for which: (i) X_i is tangent to A, B and X_{i+1} (with indices $mod\, n$); and (ii) the circles are disjoint away from tangency points, is called a *constellation* on (A, B). Prove that for any $n \geq 2$ there is some pair (A, B) and a constellation on (A, B) made of precisely n circles. Draw a picture illustrating your answer.

(c) Given an n-circle constellation $\{X_i\}$ on (A, B), prove that the tangency points $X_i \cap X_{i+1}$ for $0 \leq i \leq n-1$ all lie on a circle. Now suppose $n > 2$. Prove that if Y_0 is any circle tangent to A and B, and Y_i are constructed inductively, for $i \geq 1$, so that Y_i is tangent to A, B and Y_{i-1}, then necessarily $Y_n = Y_0$, so the chain of circles closes up to form another constellation. Does the same result hold when $n = 2$?

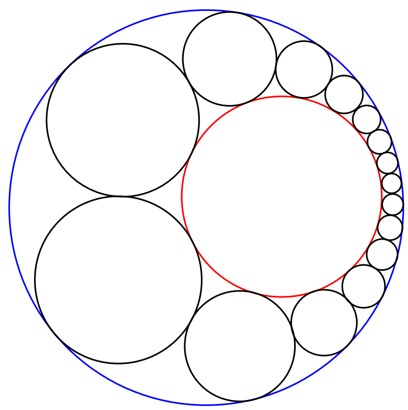

Figure 1: A constellation on the red and blue circles

E2 Let l_1, l_2, l_3 be pairwise ultraparallel hyperbolic lines whose endpoints are cyclically ordered $l_1^+, l_1^-, l_2^+, l_2^-, l_3^+, l_3^-$ at infinity $\partial\mathbb{H}^2$. Let $T_a = r_{l_2} \circ r_{l_1}$ and $T_b = r_{l_3} \circ r_{l_2}$. Prove that T_a and T_b generate a *free* subgroup of the group of orientation-preserving isometries of the hyperbolic plane. [*Hint: if U is the region bound by the l_i, and $V = U \cup r_{l_2} U$, consider a 'tiling' of the plane by copies of V.*]

E3 Let Σ be an abstract compact hyperbolic surface. Let γ_1 and γ_2 be simple *closed* geodesics on Σ, i.e. the images of smooth embeddings $\gamma_i : S^1 \to \Sigma$ which everywhere satisfy the geodesic equations. Prove that $\gamma_1 \sqcup \gamma_2$ cannot be the boundary of an embedded cylinder in Σ (i.e. a smooth subsurface homeomorphic to $S^1 \times [0, 1]$.

Construct a compact abstract hyperbolic surface Σ, and disjoint simple closed geodesics $\gamma_i \subset \Sigma$, for which $\gamma_1 \sqcup \gamma_2$ bounds an embedded subsurface Σ' of Σ homeomorphic to the complement of two disjoint discs in a torus. Can this happen if Σ has genus two? Briefly justify your answer.

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