Geometry IB – Lent 2025 – Sheet 3: *Geodesics and abstract Riemannian metrics*

1. Let $\Sigma = \{(x, y, z) \mid x^2 + y^2 = 1\}$ be the unit cylinder. Show that a geodesic on Σ through the point $(1, 0, 0)$ can be parametrized to be contained in a spiral of the form $\gamma(t) = (\cos \alpha t, \sin \alpha t, \beta t)$, where $\alpha^2 + \beta^2 = 1$.

2. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$. Suppose that a straight line $\ell = \mathbb{R} \subset \mathbb{R}^3$ lies entirely in Σ. Prove that ℓ is a geodesic on Σ. Deduce that through every point p of the hyperboloid $S = \{x^2 + y^2 = z^2 + 1\}$ there are (at least) three geodesics $\gamma_p : \mathbb{R} \to S$ defined on the entire real line $\mathbb{R}$.

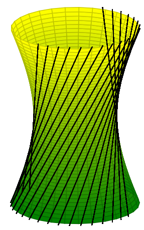

Figure 1: Lines on the hyperboloid of one sheet

3. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface of revolution in $\mathbb{R}^3$. Suppose the smooth curve $\gamma : (a, b) \to \Sigma$ satisfies the Clairaut condition
$$\rho(t) \cos \theta(t) = \text{constant}$$
where $\rho(t)$ is the distance from $\gamma(t)$ to the axis of revolution, and $\theta(t)$ is the angle between γ and the parallel at $\gamma(t)$. Suppose furthermore that there is no positive-length interval on which γ co-incides with a parallel. Show that γ is a geodesic. [*This gives a partial converse to the Clairaut relation.*]

4. (a) For $a > 0$, let Σ be the half-cone $\Sigma = \{(x, y, z) \mid z^2 = a(x^2 + y^2),\ z > 0\}$. Show that Σ is locally isometric to the Euclidean plane. By opening up the cone into a planar sector, or otherwise, show that when $a = 3$ no geodesic on Σ intersects itself, but for $a > 3$ there are geodesics which self-intersect.

   (b) Let γ be a geodesic on Σ which intersects the parallel $z = 1$ at an angle θ_0. If $\theta_0 \neq \pi_2$, show that the smallest value z_{min} of z along γ is achieved, and using Clairaut's relation show that z_{min} is independent of a. What happens when $\theta_0 = \pi/2$?

5. Given an example of a connected smooth surface $\Sigma \subset \mathbb{R}^3$ and points $p, q \in \Sigma$ for which the infimum $\inf_\gamma L(\gamma)$ of lengths of piecewise smooth curves $\gamma : [a, b] \to \Sigma$ with $\gamma(a) = p$ and $\gamma(b) = q$ is strictly smaller than the length of any piecewise smooth curve γ between p and q.

6. Let $\eta : [s_0, s_1] \to \mathbb{R}^3$ be an embedded smooth curve parametrized by arc-length. Assume that $\eta''(s) \neq 0$ for every s (i.e. η has non-zero curvature). The *binormal vector* to η is the unit vector $b(s)$ in the direction $\eta'(s) \times \eta''(s)$. Consider the ruled surface with parametrization
$$\sigma(u, v) = \eta(u) + v b(u) \qquad u \in (s_0, s_1),\ -\varepsilon < v < \varepsilon,\ \text{where } \varepsilon > 0.$$

   (i) Show that $D\sigma|_p$ is injective for $p \in \eta$, and hence is injective for all points in the image of σ with $|v|$ sufficiently small.

   (ii) Deduce that, if ε is sufficiently small, the parametrization σ defines an injective map, and that the image of σ defines a smooth surface in $\mathbb{R}^3$.

   (iii) Show that η is a geodesic on the resulting surface.

7. (a) Let $f : S^2 \to S^2$ be a diffeomorphism which is also a global isometry. By using that f sends geodesics to geodesics, or otherwise, show that f is the restriction to S^2 of an element of the orthogonal group $O(3)$.

   (b) Identifying $\mathbb{C} \cup \{\infty\}$ with $S^2 \subset \mathbb{R}^3$ via stereographic projection, prove that the Möbius group $\mathbb{P}SL(2, \mathbb{C})$ acts on (the abstract smooth surface) S^2 by diffeomorphisms. [*Hint: check that generators $z \mapsto az + b$ and $z \mapsto 1/z$ of the Möbius group act smoothly from the 'original' definition of smooth maps in terms of a smooth atlas.*]

   (c) If the Möbius map A defines an isometry of S^2, show that it commutes with the antipodal map $a : S^2 \to S^2$ (which sends $(x, y, z) \mapsto (-x, -y, -z)$).

8. (a) Define an abstract Riemannian metric on the disc $B(0,1) \subset \mathbb{R}^2$ by $\frac{du^2+dv^2}{1-u^2-v^2}$. Prove directly that diameters are then length-minimizing curves. Show that distances in the metric are bounded, but areas can be unbounded.

(b) Let $V \subset \mathbb{R}^2$ be the open square $V = \{|u| < 1, |v| < 1\}$. Define two abstract Riemannian metrics on V by

$$\frac{du^2}{(1-u^2)^2} + \frac{dv^2}{(1-v^2)^2} \qquad \text{and} \qquad \frac{du^2}{(1-v^2)^2} + \frac{dv^2}{(1-u^2)^2}.$$

(i) Define a *properly embedded path* in V to be a map $\gamma : [0, \infty) \to V$ for which the preimage of every compact set in V is compact (i.e. if $K \subset V$ is compact, $\gamma(t) \notin K$ for $t \gg 0$). Show that homeomorphisms of V take properly embedded paths to properly embedded paths.

(ii) Prove that the surfaces equipped with the given Riemannian metrics are not isometric, but there is an area-preserving diffeomorphism between them. [*Hint: for the first statement, show that exactly one of the two contains some properly embedded path of finite length.*]

The questions $E1 - E3$ are 'extras'; they may be harder or go beyond the core course material or both, but are included for the interested. They are recommended only for those already comfortable with the preceding questions (and supervisors might or might not feel inclined to talk about them).

E1 Show that the surfaces Σ and Σ' in $\mathbb{R}^3$ defined as the images of

$$\sigma(u,v) = (u\cos v, u\sin v, \ln u) \qquad \text{and} \qquad \tau(u,v) = (u\cos v, u\sin v, v)$$

(where $u > 0$ and $v > 0$) have the same Gauss curvature, but are not locally isometric.

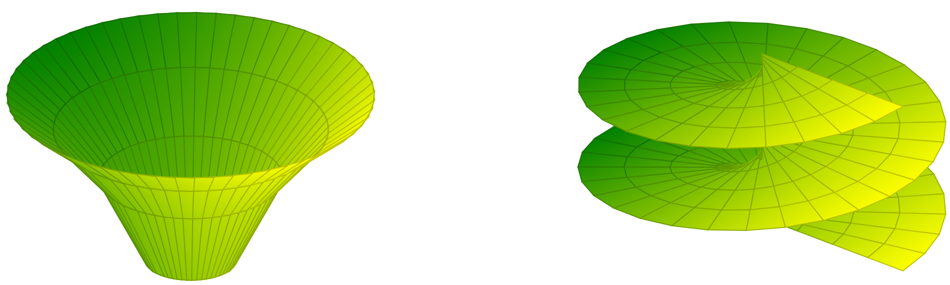

Figure 2: The surfaces Σ (left) and Σ' (right): an 'exponential cone' and a helicoid

E2 Consider a smooth surface $\Sigma \subset \mathbb{R}^3$ with Gauss curvature κ, and an allowable parametrization $\sigma : V \to U \subset \Sigma$ with first fundamental form $du^2 + G(u,v)dv^2$. Let $e = \sigma_u$, $f = \sigma_v/\sqrt{G}$ and $n = (\sigma_u \times \sigma_v)/\|\sigma_u \times \sigma_v\|$, so $\langle e, f, n\rangle$ form a 'moving frame', i.e. an orthonormal basis of $\mathbb{R}^3$ depending on the point $(u,v) \in V$.

- Show $n = e \times f$.
- Differentiating $\langle e, e\rangle = 1 = \langle f, f\rangle$ and $\langle e, f\rangle = 0$, show there are constants $\alpha, \beta, \lambda_i, \mu_j$ for which

$$e_u = \alpha \cdot f + \lambda_1 \cdot n; \quad e_v = \beta \cdot f + \mu_1 \cdot n; \quad f_u = -\alpha \cdot e + \lambda_2 \cdot n; \quad f_v = -\beta \cdot e + \mu_2 \cdot n.$$

- Show $\alpha = 0$ and $\beta = \sqrt{G}_u$.
- Show $\lambda_1\mu_2 - \lambda_2\mu_1 = e_u \cdot f_v - f_u \cdot e_v = -\beta_u = -\sqrt{G}_{uu}$.
- Recalling that the Gauss map $\mathbf{n}$ satisfies $D\mathbf{n}(\sigma_u) = n_u$ and $D\mathbf{n}(\sigma_v) = n_v$ show $n_u \times n_v = \kappa \cdot (\sigma_u \times \sigma_v)$. By considering $(n_u \times n_v).n$, conclude that $\kappa \cdot \sqrt{G} = -\sqrt{G}_{uu}$.

Deduce Gauss' *theorema egregium*: if two smooth embedded surfaces in $\mathbb{R}^3$ are isometric, they have the same Gauss curvature.

E3 The first fundamental form makes sense for a surface $\Sigma \subset \mathbb{R}^n$ for any n. Let $S^1 \subset \mathbb{C}$ be the unit circle. Let $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ be the 'product torus'. Show that the induced metric on $S^1 \times S^1$ is locally Euclidean and hence flat. Show that through any point $p \in S^1 \times S^1$ there are infinitely many closed geodesics, and also infinitely many non-closed geodesics (defined on the whole of $\mathbb{R}$). [*A closed geodesic is a geodesic σ defined on the whole of $\mathbb{R}$ but which is periodic, so for some $L > 0$ we have $\sigma(t + L) = \sigma(t)$ for every $t \in \mathbb{R}$.*]