Geometry IB – Lent 2025 – Sheet 1: *Topological and smooth surfaces*

1. (a) Prove that the cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ is not a topological surface.

   (b) Let X be the space obtained from the disjoint union $\mathbb{R}^2_a \sqcup \mathbb{R}^2_b$ of two copies of the plane $\mathbb{R}^2$, labelled by a and b, by identifying $(x, y) \in \mathbb{R}^2_a$ with $(x, y) \in \mathbb{R}^2_b$ whenever $(x, y) \neq (0, 0)$. Show that X is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

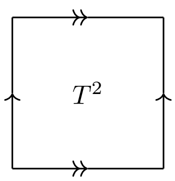
   (c) Let Σ be a topological surface, and let $\mathbb{R}_\delta$ denote the real numbers with the *discrete* topology. Show that $\Sigma \times \mathbb{R}_\delta$ is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

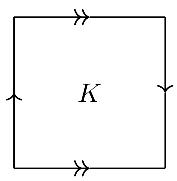
2. The torus T^2 and Klein bottle K are the topological surfaces obtained as identification spaces of a quadrilateral, as indicated in the figure.

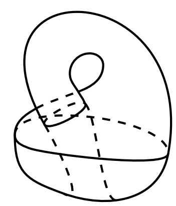
   (a) Construct a continuous surjection $p : T^2 \to T^2$ from the torus to itself such that for every $x \in T^2$, $p^{-1}(x)$ consists of exactly two points, and such that p is a 'local homeomorphism', so for every $p \in T^2$ there is an open neighbourhood $U \ni p$ for which $p|_U : U \to p(U) \subset T^2$ is a homeomorphism to its image. [Such a map is called a *covering map*.]

   (b) Show that T^2 also admits a covering map $\hat{p} : T^2 \to K$ to the Klein bottle (again with the properties that $\hat{p}^{-1}(x)$ consists of two points for each $x \in K$ and $\hat{p}$ is a local homeomorphism).

   (c) Give three pairwise non-conjugate subgroups $\mathbb{Z}/2 \leq \mathrm{Homeo}(T^2)$ of the group, under composition of maps, of homeomorphisms from T^2 to itself.







3. Draw inside the identification square for the Klein bottle K the set of points in which the 'usual' map $f : K \to \mathbb{R}^3$ (as drawn on the right above) fails to be injective. By considering a map $h = (f, \eta) : K \to \mathbb{R}^3 \times \mathbb{R}$, where η is a function of the width co-ordinate of the square, explain why K can be continuously embedded in $\mathbb{R}^4$, i.e. there is a continuous map $K \to \mathbb{R}^4$ which is a homeomorphism to its image. [*You do not need an explicit formula for f. Many helpful pictures can be found at* `https://en.wikipedia.org/wiki/Klein_bottle`.]

4. (a) Consider a decomposition of a topological surface into polygons (with V vertices, E edges and F faces), where all vertices have valence ≥ 3 and every face contains ≥ 3 vertices. Let F_n denote the number of faces bound by precisely n edges, and V_m the number of vertices where precisely m edges meet. Show that $\sum_n n\, F_n = 2E = \sum_m m\, V_m$. If $V_3 = 0$, deduce $E \geq 2V$, whilst if $F_3 = 0$ deduce $E \geq 2F$. If the surface is a sphere, deduce that $V_3 + F_3 > 0$.

   (b) For a polygonal decomposition of a sphere, further show that

   $$\sum_n (6 - n) F_n = 12 + 2 \sum_m (m - 3) V_m.$$

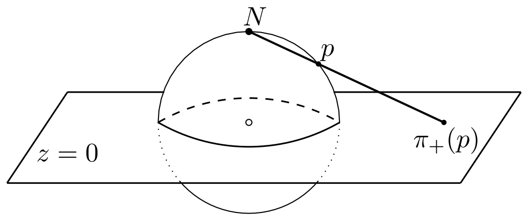
   If each face has at least 3 edges and at least 3 edges meet at each vertex, deduce that $3F_3 + 2F_4 + F_5 \geq 12$.

   (c) The surface of a football is decomposed into hexagons and pentagons, with precisely 3 faces meeting at each vertex. How many pentagons are there?

5. (a) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $\pi_\pm : S^2 \backslash \{(0, 0, \pm 1)\} \to \mathbb{R}^2 = \{z = 0\}$ denote stereographic projection from the north / south poles $(0, 0, \pm 1) \in S^2$. Show that the transition function between these two charts is given by $(u, v) \mapsto (u/(u^2 + v^2), v/(u^2 + v^2))$. Deduce that this atlas gives S^2 the structure of an abstract smooth surface.

   (b) Now consider the chart on S^2 given by (U, ϕ) where $U = \{y < 0\}$ and $\phi : U \to \mathbb{R}^2$ maps $(x, y, z) \to (x, z)$. What is the image of ϕ? Check explicitly that this chart is *compatible* with the smooth atlas defined by $\pi_\pm$, i.e. the transition functions for the atlas with charts $\{\pi_+, \pi_-, \phi\}$ are all smooth.

   (c) Let $a : S^2 \to S^2$ denote the antipodal map, $x \mapsto -x$. Show that a is a diffeomorphism of the abstract smooth surface S^2, and deduce that the real projective plane $\mathbb{RP}^2$ also admits the structure of an abstract smooth surface.



6. A *smooth curve* in $\mathbb{R}^3$ is a subspace which can locally be given as the image of a smooth injective map $(-1,1) \to \mathbb{R}^3$ with injective differential. Let $H : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Suppose the level sets $H^{-1}(0)$ and $K^{-1}(0)$ are smooth surfaces in $\mathbb{R}^3$ which intersect along a smooth curve γ in $\mathbb{R}^3$. Suppose $p = (x_0, y_0, z_0)$ belongs to γ, and that at p the expression $K_y H_z - K_z H_y$ is non-vanishing. Show that in some neighbourhood of p the curve γ can be parametrized by the variable x, and that if we write $\gamma(x) = (x, y(x), z(x))$ then

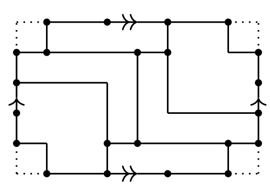
$$y'(x) = (K_z H_x - K_x H_z)/(K_y H_z - K_z H_y).$$

7. Consider the subspace $\Sigma \subset \mathbb{R}^3$ defined by $\{(x,y,z) \in \mathbb{R}^3 \,:\, z = x^3 + y^3 - 3xy\}$. Observe that Σ is a smooth surface in $\mathbb{R}^3$ (*why?*). Show that the level set $\Sigma \cap \{z = c\}$ is a (not necessarily connected) smooth curve in the plane unless $c \in \{-1, 0\}$. Sketch the level sets for values $c = -1$ and $c = 0$. [*For $c = -1$, it may help to factorise $x^3 + y^3 - 3xy + 1 = (x+y+1)(x^2+y^2-xy-x-y+1)$.*]

The questions $E1 - E3$ are 'extras'; they may be harder or go beyond the core course material or both, but are included for the interested. Supervisors might or might not feel inclined to talk about them.

E1 (a) View a polygonal decomposition of a topological surface S with all vertices of valence ≥ 3 as a map drawn on S. The map can be coloured with N colours if we can colour faces so that faces which share an edge have different colours (but faces which meet only at a vertex may have the same colour). Suppose that $N \in \mathbb{N}$ is a positive integer such that $2E/F < N$ for every possible decomposition of S. Show that every map on S can be coloured with at most N colours. [*Hint: Induct on F, noting that the result is straightforward if $F < N$.*]

(b) Prove that for any decomposition of S (where all vertices have valence at least 3), we have $2E/F \leq 6\,(1 - \chi(S)/F)$, where $\chi(S)$ is the Euler characteristic of S. Deduce that any map on a torus can be coloured with 7 colours. The map on the torus depicted below can be coloured with 7 colours but not fewer – why? [*The black dots represent the vertices; note the 'corner' of the rectangle is not one.*]



E2 (a) Prove that the set of straight lines in $\mathbb{R}^2$ admits the structure of a topological surface Σ which is homeomorphic to a Möbius band. [One possibility is to define an atlas with two charts by considering separately subsets of non-horizontal and non-vertical lines.]

(b) Prove that the group of affine transformations

$$\mathrm{Aff}(\mathbb{R}^2) = \{x \mapsto Ax + b \,|\, A \in GL(2;\mathbb{R}),\ b \in \mathbb{R}^2\}$$

acts on Σ by homeomorphisms.

(c) Prove that there is no metric (in the sense of metric spaces) $d : \Sigma \times \Sigma \to \mathbb{R}_{\geq 0}$ on Σ which induces the given topology on Σ and for which $\mathrm{Aff}(\mathbb{R}^2)$ acts by isometries on (Σ, d). [*Hint: show that the distance between two lines depends only on the angle between them, and then think about a pair of distinct but parallel lines.*]

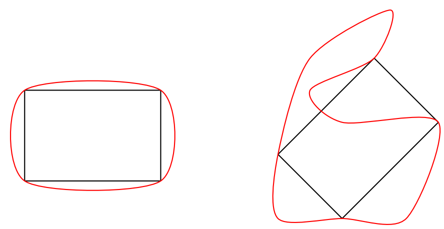
E3 (a) Let X be the set of unordered pairs of points on a circle S^1. Explain why X is naturally a quotient space of the torus T^2. By considering T^2 as a quotient space of a square, or otherwise, show that X is homeomorphic to $\mathbb{RP}^2 \backslash \mathring{D}$, the complement of an open disc $\mathring{D} \subset \mathbb{RP}^2$ in $\mathbb{RP}^2$.

(b) Let $\phi : D = \{|z| \leq 1\} \to \mathbb{R}^2$ be a continuous injection of a closed disc D, with boundary $C = \phi(S^1) \subset \mathbb{R}^2$. Define a map $f : C \times C \to \mathbb{R}^3$ via

$$(u, v) \mapsto \left(\frac{u+v}{2}, |u-v| \right) \in \mathbb{R}^2 \times \mathbb{R}.$$

Show that f defines a map on X, which extends to a continuous map $\hat{\phi} : \mathbb{RP}^2 \to \mathbb{R}^3$.

(c) Using without proof that a compact non-orientable topological surface cannot be continuously embedded in $\mathbb{R}^3$, deduce that C bounds an *inscribed rectangle*, i.e. there are four pairwise-distinct points on the curve which form the vertices of a rectangle in $\mathbb{R}^2$ (the rectangle need not be wholly contained in $\phi(D)$, cf. figure below).



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