IB GEOMETRY EXAMPLES 4

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Show that a non-identity Möbius transformation T has exactly one or two fixed points in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Show that if T corresponds, under stereographic projection, to a rotation of S^2, then it has two fixed points z_i which satisfy $z_2 = -1/\bar{z}_1$. If $T \in$ Möb has two fixed points z_i and $z_2 = -1/\bar{z}_1$, prove that either T corresponds to a rotation, or one of the two fixed points (say z_1) is attractive, i.e. $T^n(z) \to z_1$ for all $z \neq z_2$ as $n \to \infty$.

2. (a) Show that inversion in the circle $\{|z-a| = r\}$ is given by $z \mapsto a + \frac{r^2}{\bar{z}-\bar{a}}$.
(b) Show that every Möbius map is a composition of inversions.

3. Show from first principles that a vertical line segment is length-minimizing and hence defines a geodesic in the hyperbolic upper half-plane, i.e. the upper half-plane with the abstract metric $\frac{dx^2+dy^2}{y^2}$.

4. Let z_1, z_2 be distinct points in the upper half plane $\mathfrak{h}$. Suppose that the hyperbolic line through z_1 and z_2 meets the real axis at points w_1 and w_2, where z_1 lies on the hyperbolic line segment $w_1 z_2$ (and where one w_i may be ∞). Show that the hyperbolic distance $d_{hyp}(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2 taken in an appropriate order

5. (a) Let $P \in S^2 \subset \mathbb{R}^3$ be a point on the round sphere. The spherical circle with centre P and radius ρ is the set $\{w \in S^2 \,|\, d_{sph}(w, P) = \rho\}$, where d_{sph} is the spherical metric (induced by the first fundamental form of the embedding). Prove that a spherical circle of radius ρ is a Euclidean circle. Prove that its circumference is $2\pi \sin(\rho)$ and that it bounds a disc on S^2 of area $2\pi(1-\cos(\rho))$.
(b) Let $C \subset \mathfrak{h}$ be a hyperbolic circle with centre $p \in \mathfrak{h}$ and radius ρ, i.e. the locus $\{w \in \mathfrak{h} \,|\, d_{hyp}(w, p) = \rho\}$ for some $\rho > 0$. Show that C is a Euclidean circle. If $p = ic$ for $c \in \mathbb{R}_{>0}$, find the centre and radius of C as a Euclidean circle. Show the hyperbolic circumference of C is $2\pi \sinh(\rho)$, and the hyperbolic area of the disc it bounds is $2\pi(\cosh(\rho) - 1)$. Deduce that no hyperbolic triangle contains a hyperbolic circle of radius $> \cosh^{-1}(3/2)$.
(c) Show that there is some $\delta > 0$ such that, in any hyperbolic triangle, the union of the δ-neighbourhoods of two of the sides completely contains the 3rd side. (Does such a δ exist for triangles in the Euclidean plane?)

6. Fix a hyperbolic triangle $\Delta \subset \mathbb{H}^2$ with interior angles A, B, C and side lengths (in the hyperbolic metric) a, b, c, where a is the side opposite the vertex with angle A, etc.
(a) Suppose that C is a right-angle. By applying the 'hyperbolic cosine' formula in two different ways, prove that

$$\sin(A)\sinh(c) = \sinh(a).$$

(b) Deduce that for a general hyperbolic triangle, one has

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$$

7. (a) Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the common perpendicular is unique. Show that, up to isometry, for $t > 0$ there is a unique configuration of ultraparallel lines for which the segment of the common perpendicular between the lines has length t.

(b) Let l_1, l_2 be ultraparallel hyperbolic lines, and let r_{l_i} denote the hyperbolic isometry given by reflection in l_i. Prove that $r_{l_1} \circ r_{l_2}$ has infinite order.

8. (a) Consider the 'ideal' hyperbolic square with vertices at $0, 1, \infty, -1$ in the upper half-plane model. By gluing the edges of the square by isometries, or otherwise, prove that there is a *complete* hyperbolic metric on the smooth surface $S^2 \backslash \{p, q, r\}$ given by the complement of 3 distinct points in the sphere.

(b) Construct a non-orientable compact hyperbolic surface.

9. Consider stereographic projection from the north pole $\pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$. Note that it defines a diffeomorphism from the southern hemisphere $S^2_- = \{(x, y, z) \in S^2 : \; z < 0\}$ to the unit disc $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 < 1\}$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection $p(x, y, z) = (x, y)$. Show that the diffeomorphism

$$f := p \circ \pi_+^{-1} : \mathbb{D} \to \mathbb{D}$$

has the expression

$$f(\xi) = \frac{2\xi}{|\xi|^2 + 1},$$

where $\xi = x + iy$. The Beltrami-Klein model of hyperbolic geometry is obtained by equipping $\mathbb{D}$ with the Riemannian metric g_K such that $f : (\mathbb{D}, g_{hyp}) \to (\mathbb{D}, g_K)$ is an isometry, where g_{hyp} is the hyperbolic metric in the disk. Show that hyperbolic lines in the Beltrami-Klein model are the (Euclidean) straight line segments contained in the disc $\mathbb{D}$. (You may use that π_+ is a conformal map that takes circles on the sphere to lines and circles in the plane.)

10. Let Σ be an abstract compact hyperbolic surface. Let γ_1 and γ_2 be simple *closed* geodesics on Σ, i.e. the images of smooth embeddings $\gamma_i : S^1 \to \Sigma$ which everywhere satisfy the geodesic equations. Prove that the disjoint union $\gamma_1 \sqcup \gamma_2$ cannot be the boundary of an embedded cylinder in Σ (i.e. a smooth subsurface homeomorphic to $S^1 \times [0, 1]$.

Construct a compact abstract hyperbolic surface Σ, and disjoint simple closed geodesics $\gamma_i \subset \Sigma$, for which $\gamma_1 \sqcup \gamma_2$ bounds an embedded subsurface Σ' of Σ homeomorphic to the complement of two disjoint discs in a torus. Can this happen if Σ has genus two? Briefly justify your answer.