IB GEOMETRY EXAMPLES 3

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let $\Sigma = \{(x,y,z) \,|\, x^2+y^2=1\}$ be the unit cylinder. Show that a geodesic on Σ through the point $(1,0,0)$ can be parametrized to be contained in a spiral of the form $\gamma(t) = (\cos\alpha t, \sin\alpha t, \beta t)$, where $\alpha^2+\beta^2=1$.

2. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$. Suppose that a straight line $\ell = \mathbb{R} \subset \mathbb{R}^3$ lies entirely in Σ. Prove that ℓ is a geodesic on Σ. Deduce that through every point p of the hyperboloid $S = \{x^2+y^2=z^2+1\}$ there are (at least) three geodesics $\gamma_p : \mathbb{R} \to S$ defined on the entire real line $\mathbb{R}$.

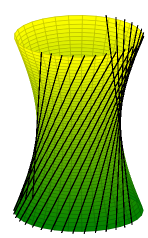

FIGURE 1. Lines on the hyperboloid of one sheet

3. (a) For $a>0$, let Σ be the half-cone $\Sigma = \{(x,y,z) \,|\, z^2 = a(x^2+y^2),\ z>0\}$. Show that Σ is locally isometric to the Euclidean plane. By opening up the cone into a planar sector, or otherwise, show that when $a=3$ no geodesic on Σ intersects itself, but for $a>3$ there are geodesics which self-intersect.

(b) Let γ be a geodesic on Σ which intersects the parallel $z=1$ at an angle θ_0. If $\theta_0 \neq \pi/2$, show that the smallest value z_{min} of z along γ is achieved, and using Clairaut's relation show that z_{min} is independent of a. What happens when $\theta_0 = \pi/2$?

4. Given an example of a connected smooth surface $\Sigma \subset \mathbb{R}^3$ and points $p,q \in \Sigma$ for which the infimum $\inf_\gamma L(\gamma)$ of lengths of piecewise smooth curves $\gamma : [a,b] \to \Sigma$ with $\gamma(a)=p$ and $\gamma(b)=q$ is strictly smaller than the length of any piecewise smooth curve γ between p and q.

5. Let $\eta : [s_0,s_1] \to \mathbb{R}^3$ be an embedded smooth curve parametrized by arc-length. Assume that $\eta''(s) \neq 0$ for every s (i.e. η has non-zero curvature). The *binormal vector* to η is the unit vector $b(s)$ in the direction $\eta'(s) \times \eta''(s)$. Consider the ruled surface with parametrization

$$\sigma(u,v) = \eta(u) + vb(u) \qquad u \in (s_0,s_1),\ -\varepsilon < v < \varepsilon,\ \text{where } \varepsilon > 0.$$

(a) Show that $D\sigma|_{(u,0)}$ is injective for all u, and hence $D\sigma|_{(u,v)}$ is injective for all (u,v) with $|v|$ sufficiently small.

(b) Deduce that, if ε is sufficiently small, the parametrization σ defines an injective map, and that the image of σ defines a smooth surface in $\mathbb{R}^3$.

(c) Show that η is a geodesic on the resulting surface.

6. Let Σ_1 and Σ_2 be two smooth surfaces in $\mathbb{R}^3$. Show that an isometry $f : \Sigma_1 \to \Sigma_2$ maps geodesics to geodesics.

7. (a) Let $f : S^2 \to S^2$ be an isometry. By using that f sends geodesics to geodesics, or otherwise, show that f is the restriction to S^2 of an element of the orthogonal group $O(3)$.

(b) Identifying $\mathbb{C} \cup \{\infty\}$ with $S^2 \subset \mathbb{R}^3$ via stereographic projection, prove that the Möbius group $\mathbb{P}SL(2,\mathbb{C})$ acts on (the abstract smooth surface) S^2 by diffeomorphisms. [*Hint: check that generators* $z \mapsto az + b$ *and* $z \mapsto 1/z$ *of the Möbius group act smoothly from the 'original' definition of smooth maps in terms of a smooth atlas.*]

(c) If the Möbius map A defines an isometry of S^2, show that it commutes with the antipodal map $a : S^2 \to S^2$ (which sends $(x,y,z) \mapsto (-x,-y,-z)$). (One can also show that any Möbius map commuting with the antipodal map is an isometry.)

8. Define an abstract Riemannian metric on the disc $B(0,1) \subset \mathbb{R}^2$ by $\frac{du^2+dv^2}{1-u^2-v^2}$. Prove directly that diameters are length-minimizing curves. Show that distances in the metric are bounded, but areas can be unbounded.

9. Let $V \subset \mathbb{R}^2$ be the open square $V = \{|u| < 1,\, |v| < 1\}$. Define two abstract Riemannian metrics on V by

$$\frac{du^2}{(1-u^2)^2} + \frac{dv^2}{(1-v^2)^2} \qquad \text{and} \qquad \frac{du^2}{(1-v^2)^2} + \frac{dv^2}{(1-u^2)^2}.$$

(a) Define a *properly embedded path* in V to be a map $\gamma : [0,\infty) \to V$ for which the preimage of every compact set in V is compact (i.e. if $K \subset V$ is compact, $\gamma(t) \notin K$ for $t \gg 0$). Show that homeomorphisms of V take properly embedded paths to properly embedded paths.

(b) Prove that the surfaces equipped with the given Riemannian metrics are not isometric, but there is an area-preserving diffeomorphism between them. [*Hint: for the first statement, show that exactly one of the two contains some properly embedded path of finite length.*]

10. Let $V \subset \mathbb{R}^2$ be an open set equipped with an abstract Riemannian metric $\rho(du^2 + dv^2)$, where $\rho : V \to \mathbb{R}$ is a positive smooth function. Let $H : V \times \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$H(u,v,p_u,p_v) = \frac{1}{2\rho}(p_u^2 + p_v^2).$$

Show that $(u(t),v(t))$ is a geodesic if and only if the curve $(u(t),v(t),p_u(t),p_v(t))$, satisfies Hamilton's equations:

$$\dot{u} = \frac{\partial H}{\partial p_u}, \quad \dot{v} = \frac{\partial H}{\partial p_v}, \quad \dot{p}_u = -\frac{\partial H}{\partial u}, \quad \dot{p}_v = -\frac{\partial H}{\partial v},$$

where $p_u(t) = \rho(u(t),v(t))\dot{u}(t)$ and $p_v(t) = \rho(u(t),v(t))\dot{v}(t)$.