IB GEOMETRY EXAMPLES 1

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Show that a topological surface is connected if and only if it is path-connected.

2. (a) Prove that the cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ is not a topological surface.

(b) Let X be the space obtained from the disjoint union $\mathbb{R}^2_a \sqcup \mathbb{R}^2_b$ of two copies of the plane $\mathbb{R}^2$, labelled by a and b, by identifying $(x, y) \in \mathbb{R}^2_a$ with $(x, y) \in \mathbb{R}^2_b$ whenever $(x, y) \neq (0, 0)$. Show that X is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

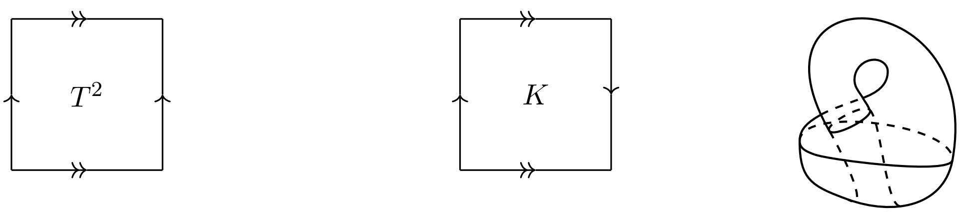
(c) Let Σ be a topological surface, and let $\mathbb{R}_\delta$ denote the real numbers with the *discrete* topology. Show that $\Sigma \times \mathbb{R}_\delta$ is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

3. The torus T^2 and Klein bottle K are the topological surfaces obtained as identification spaces of a quadrilateral, as indicated in the figure.

(a) Construct a continuous surjection $p : T^2 \to T^2$ from the torus to itself such that for every $x \in T^2$, $p^{-1}(x)$ consists of exactly two points, and such that p is a 'local homeomorphism', so for every $p \in T^2$ there is an open neighbourhood $U \ni p$ for which $p|_U : U \to p(U) \subset T^2$ is a homeomorphism to its image. [Such a map is called a *covering map*.]

(b) Show that T^2 also admits a covering map $\hat{p} : T^2 \to K$ to the Klein bottle (again with the properties that $\hat{p}^{-1}(x)$ consists of two points for each $x \in K$ and $\hat{p}$ is a local homeomorphism).

(c) Give three pairwise non-conjugate subgroups $\mathbb{Z}_2 \leq \mathrm{Homeo}(T^2)$ of the group, under composition of maps, of homeomorphisms from T^2 to itself.



4. Draw inside the identification square for the Klein bottle K the set of points in which the 'usual' map $f : K \to \mathbb{R}^3$ (as drawn on the right above) fails to be injective. By considering a map $h = (f, \eta) : K \to \mathbb{R}^3 \times \mathbb{R}$, where η is a function of the width co-ordinate of the square, explain why K can be continuously embedded in $\mathbb{R}^4$, i.e. there is a continuous map $K \to \mathbb{R}^4$ which is a homeomorphism to its image. [*You do not need an explicit formula for f. Many helpful pictures can be found at* `https://en.wikipedia.org/wiki/Klein_bottle`.]

5. (a) Consider a decomposition of a topological surface into polygons (with V vertices, E edges and F faces), where all vertices have valence ≥ 3 and every face contains ≥ 3 vertices (and every edge is an edge of exactly two faces). Let F_n denote the number of faces bound by precisely n edges, and V_m the number of vertices where precisely m edges meet. Show that $\sum_n n\, F_n = 2E = \sum_m m\, V_m$. If $V_3 = 0$, deduce $E \geq 2V$, whilst if $F_3 = 0$ deduce $E \geq 2F$. If the surface is a sphere, deduce that $V_3 + F_3 > 0$.

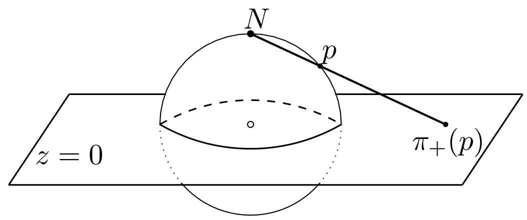
(b) For a polygonal decomposition of a sphere, further show that

$$\sum_n (6 - n)F_n = 12 + 2\sum_m (m - 3)V_m.$$

If each face has at least 3 edges and at least 3 edges meet at each vertex, deduce that $3F_3 + 2F_4 + F_5 \geq 12$.

(c) The surface of a football is decomposed into hexagons and pentagons, with precisely 3 faces meeting at each vertex. How many pentagons are there?

6. (a) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $\pi_\pm : S^2 \backslash \{(0,0,\pm 1)\} \to \mathbb{R}^2 = \{z=0\}$ denote stereographic projection from the north / south poles $(0,0,\pm 1) \in S^2$. Show that the transition function between these two charts is given by $(u,v) \mapsto (u/(u^2+v^2), v/(u^2+v^2))$. Deduce that this atlas gives S^2 the structure of an abstract smooth surface.



(b) Now consider the chart on S^2 given by (U, ϕ) where $U = \{y < 0\}$ and $\phi : U \to \mathbb{R}^2$ maps $(x,y,z) \to (x,z)$. What is the image of ϕ? Check explicitly that this chart is *compatible* with the smooth atlas defined by $\pi_\pm$, i.e. the transition functions for the atlas with charts $\{\pi_+, \pi_-, \phi\}$ are all smooth.

(c) Let $a : S^2 \to S^2$ denote the antipodal map, $x \mapsto -x$. Show that a is a diffeomorphism of the abstract smooth surface S^2, and deduce that the real projective plane $\mathbb{RP}^2$ also admits the structure of an abstract smooth surface.

7. A *smooth curve* in $\mathbb{R}^3$ is a subset which can locally be given as the image of a smooth injective map $(-1,1) \to \mathbb{R}^3$ with injective differential. Let $H : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Suppose the level sets $H^{-1}(0)$ and $K^{-1}(0)$ are smooth surfaces in $\mathbb{R}^3$ which intersect along a smooth curve γ in $\mathbb{R}^3$. Suppose $p = (x_0, y_0, z_0)$ belongs to γ, and that at p the expression $K_y H_z - K_z H_y$ is non-vanishing. Show that in some neighbourhood of p the curve γ can be parametrized by the variable x, and that if we write $\gamma(x) = (x, y(x), z(x))$ then

$$y'(x) = (K_z H_x - K_x H_z)/(K_y H_z - K_z H_y).$$

Verify this formula in the case $H = x^2 + y^2 - z$ and $K = x^2 + y^2 + z^2 - 1$.

8. Consider the subset $\Sigma \subset \mathbb{R}^3$ defined by $\{(x,y,z) \in \mathbb{R}^3 : z = x^3 + y^3 - 3xy\}$. Observe that Σ is a smooth surface in $\mathbb{R}^3$ (*why?*). Show that the level set $\Sigma \cap \{z = c\}$ is a (not necessarily connected) smooth curve in the plane unless $c \in \{-1, 0\}$. Sketch the level sets for values $c = -1$ and $c = 0$. [*For $c = -1$, it may help to factorise* $x^3 + y^3 - 3xy + 1 = (x+y+1)(x^2+y^2-xy-x-y+1)$.]

9. Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^3$ be two smooth surfaces and let $f : \Sigma_1 \to \Sigma_2$ be a smooth map between them. Given $p \in \Sigma_1$ consider allowable parametrisations $\sigma : U \to \Sigma_1$, $\tau : V \to \Sigma_2$ so that $\sigma(0) = p$ and $\tau(0) = f(p)$. Define a linear map $Df|_p : T_p\Sigma_1 \to T_{f(p)}\Sigma_2$ by

$$Df|_p := D\tau|_0 \circ D(\tau^{-1} \circ f \circ \sigma)|_0 \circ (D\sigma|_0)^{-1}.$$

Show that this definition is independent of the choices of allowable parametrisations. Show that if $Df|_p$ is a linear isomorphism, then f is a local diffeomorphism at p.

10. Consider stereographic projection from the north pole $\pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ as in Question 6. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $P : \mathbb{C} \to \mathbb{C}$ be a non constant complex polynomial. Define $F : S^2 \to S^2$ by

$$F(p) = \pi_+^{-1} \circ P \circ \pi_+(p), \text{ for } p \in S^2 \setminus \{(0,0,1)\}, \quad F(0,0,1) = (0,0,1).$$

Show that F is smooth. For $P(\zeta) = \zeta^3 + \zeta^2 + 1$, find the points $p \in S^2$ for which $DF|_p$ fails to be a linear isomorphism.