Geometry IB – Lent 2022 – Sheet 2: *Smooth surfaces in* $\mathbb{R}^3$

1. (a) Let (X, d) be a metric space. If $X = \{p_1, p_2, p_3\}$ has only 3 points, show that X admits an isometric embedding into the Euclidean plane $(\mathbb{R}^2, d_{eucl})$ (i.e. a continuous injection preserving distances).

   (b) Show the four-point space $X = \{p_1, p_2, p_3, q\}$ with distances
   $$d(p_i, p_j) = 2L = d(p_1, q); \quad d(p_2, q) = L = d(p_3, q)$$
   is a metric space which admits no isometric embedding into Euclidean space $\mathbb{R}^n$ for any n.

   (c) Let $L = \pi/4$ in (b). Show that X isometrically embeds in the unit sphere $S^2 \subset \mathbb{R}^3$, and deduce that no subset of S^2 containing an open hemisphere admits an isometric embedding into the plane. [*With more work one can show by similar 'elementary' methods that no open subset of S^2 isometrically embeds into* $(\mathbb{R}^2, d_{eucl})$.]

2. Show that each of the following parametrizations $\sigma : U \to \mathbb{R}^3$ is allowable, find the first fundamental form and sketch the image of σ.

   (a) $U = \{(u, v) \in \mathbb{R}^2 \,|\, u > v\}, \quad \sigma(u, v) = (u + v, 2uv, u^2 + v^2)$;

   (b) $U = \{(r, z) \in \mathbb{R}^2 \,|\, r > 0\}, \quad \sigma(r, z) = (r \cos(z), r \sin(z), z)$.

3. *Mercator's projection* of the sphere is the chart whose inverse is the local parametrization
   $$\sigma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$
   Prove that this determines an allowable chart on the complement of a longitude, which sends lines of longitude and latitude to straight lines in the plane, and which preserves angles but not areas (*cf. Greenland versus Africa on a map; see also* `https://en.wikipedia.org/wiki/Mercator_projection`).

4. (a) Place the unit sphere $S^2 \subset \mathbb{R}^3$ inside a vertical circular cylinder C of radius one. Prove that horizontal projection from S^2 to C preserves area. Deduce that S^2 admits a smooth atlas of charts which are area-preserving.

   (b) A *lune* is one component of the region on the unit sphere S^2 cut out by two great circles (so it is a bigon). Prove that if the lune has internal angle α, it has area 2α. Hence, or otherwise, prove that a *spherical triangle*, i.e. a connected region bound by 3 great circles and with internal angles α, β, γ each less than π, has area $\alpha + \beta + \gamma - \pi$.

5. (a) Let $\eta : (a, b) \to \mathbb{R}^3$ be a smooth curve given by $\eta(t) = (f(t), 0, g(t))$. Suppose η' is never zero, η is a homeomorphism to its image and $f(u) > 0$ for all u. Let Σ_η denote the associated surface of revolution given by rotating η around the z-axis. Prove that the Gauss curvature κ of Σ_η is given by
   $$\kappa = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}.$$
   If η is parametrized by arc-length, show $\kappa = -f''/f$.

   (b) Calculate κ for the hyperboloid of one sheet $\{x^2 + y^2 = z^2 + 1\}$ and of two sheets $\{x^2 + y^2 = z^2 - 1\}$. Describe the qualitative properties of κ (its sign, its behaviour near infinity). Illustrate the results with a picture.

6. Let Σ_η be as in the previous question. Let $n : \Sigma_\eta \to S^2$ be the Gauss map. Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation by angle θ about the z-axis. Prove that for $p \in \Sigma_\eta$, $n(R_\theta(p)) = R_\theta(n(p))$.

   (d) Hence, or otherwise, prove that for the hyperboloid $\{x^2 + y^2 = z^2 + 1\}$, the image of the Gauss map is the open annulus $\{|z| < 1/\sqrt{2}\} \subset S^2$.

7. Let $T \subset \mathbb{R}^3$ be the smooth embedded torus obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane around the z axis. Sketch T, and draw an illustration of where on T the Gauss curvature κ is positive, negative and zero.

8. Consider the surface $\Sigma \subset \mathbb{R}^3$ with parametrization
   $$\sigma(u, v) = \gamma(u) + v\, a(u) \qquad u \in [0, 2\pi), v \in (-1, 1)$$
   where $\gamma(u) = (\cos u, \sin u, 0)$ and $a(u) = (\cos(u/2) \cos(u), \cos(u/2) \sin(u), \sin(u/2))$. (This is an example of a 'ruled surface': one which is locally swept out by a moving Euclidean straight line.) Sketch Σ, and prove that Σ is a smooth surface for which the Gauss curvature κ is everywhere negative.

The questions $E1 - E3$ are 'extras'; they may be harder or go beyond the core course material or both, but are included for the interested. They are recommended only for those already comfortable with the preceding questions (and supervisors might or might not feel inclined to talk about them).

E1 A compact smooth surface $\Sigma \subset \mathbb{R}^3$ is *strictly convex* if it is the boundary of a closed region $R \subset \mathbb{R}^3$ with the property that for any $x, y \in R$, the straight line segment $[x, y] \subset R$, and $[x, y] \cap \Sigma \subset \{x, y\}$. Show that the Gauss map of a strictly convex surface Σ is a bijection. If, furthermore, the Gauss curvature of Σ is everywhere positive, show that the Gauss map is a diffeomorphism, and deduce that $\int_\Sigma \kappa \, dA = 4\pi$, where κ is the Gauss curvature and dA is the area form on Σ. [*This is the 'convex' Gauss-Bonnet theorem.*]

E2 Let $\Sigma \subset \mathbb{R}^3$ be a smooth oriented surface, and $p \in \Sigma$. Let $n(p)$ be the unit normal vector at p. Let $v \in T_p\Sigma$ be a unit vector (with respect to the first fundamental form). Let γ_v be the plane curve which is the intersection of Σ and the affine two-plane $\mathbb{R}^2 = p + \mathrm{Span}\langle v, n(p)\rangle$. Viewing the second fundamental form as a bilinear form II_p on $T_p\Sigma$, show that $II_p(v, v)$ is the curvature of the plane curve γ_v at p.

[*The curvature of a plane curve* $\gamma : (a, b) \to \mathbb{R}^2$ *parametrized by arc-length is the function* $\kappa : (a, b) \to \mathbb{R}$ *for which* $\gamma''(s) = \kappa(s) n_\gamma(s)$, *with* $n_\gamma(s)$ *the unit normal to* γ *for which* $\langle \gamma'(s), n_\gamma(s)\rangle$ *forms a positively oriented basis for* $\mathbb{R}^2$.]

E3 (a) Let $\sigma : V \to \Sigma \subset \mathbb{R}^3$ be an allowable parametrization for a subset of a smooth surface in $\mathbb{R}^3$, with $V \subset \mathbb{R}^2$ a connected open subset with co-ordinate (u, v). Let $\alpha_{ijk} = \langle \sigma_i, \sigma_{jk}\rangle$ where $i, j, k \in \{u, v\}$ and subscripts denote partial differentiation. Show the functions α_{ijk} on V are determined by the first fundamental form.

(b) Let P be the matrix with columns σ_u, σ_v, n where n denotes the unit normal to Σ. Consider the matrix-valued functions on V defined by

$$C = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; D_1 = \begin{pmatrix} \alpha_{uuu} & \alpha_{uvu} & -L \\ \alpha_{vuu} & \alpha_{vvu} & -M \\ L & M & 0 \end{pmatrix}; \; D_2 = \begin{pmatrix} \alpha_{uuv} & \alpha_{uvv} & -M \\ \alpha_{vuv} & \alpha_{vvv} & -N \\ M & N & 0 \end{pmatrix}.$$

Show $P^t.P = C$, $P^t.P_u = D_1$ and $P^t.P_v = D_2$.

(c) Letting $A = C^{-1}D_1$ and $B = C^{-1}D_2$, deduce that each row ξ of P satisfies a system of linear first order partial differential equations

$$\xi_u = A^t\xi; \quad \xi_v = B^t\xi.$$

Assuming the uniqueness of solutions to such a differential system with a given initial condition, deduce that a connected smooth surface in $\mathbb{R}^3$ is determined up to rigid motion by its first and second fundamental forms.

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