Geometry IB – 2020/21 – Sheet 4: *Hyperbolic surfaces and Gauss-Bonnet* [Circa Lectures 19–24]

1. Show that a non-identity Möbius transformation T has exactly one or two fixed points in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Show that if T corresponds, under stereographic projection, to a rotation of S^2, then it has two fixed points z_i which satisfy $z_2 = -1/\bar{z}_1$. If $T \in$ Möb has two fixed points z_i and $z_2 = -1/\bar{z}_1$, prove that either T corresponds to a rotation, or one of the two fixed points (say z_1) is attractive, i.e. $T^n(z) \to z_1$ for all $z \neq z_2$ as $n \to \infty$.

2. Show that inversion in the circle $\{|z - a| = r\}$ is given by $z \mapsto a + \frac{r^2}{\bar{z} - \bar{a}}$.

3. (a) Let A, B be disjoint circles in $\mathbb{C}$. Show that there is a Möbius transformation which takes A and B to two concentric circles.

   (b) A collection of circles $X_i \subset \mathbb{C}$, $0 \leq i \leq n-1$, for which: (i) X_i is tangent to A, B and X_{i+1} (with indices $mod\, n$); and (ii) the circles are disjoint away from tangency points, is called a *constellation* on (A, B). Prove that for any $n \geq 2$ there is some pair (A, B) and a constellation on (A, B) made of precisely n circles. Draw a picture illustrating your answer.

   (c) Given an n-circle constellation $\{X_i\}$ on (A, B), prove that the tangency points $X_i \cap X_{i+1}$ for $0 \leq i \leq n-1$ all lie on a circle. Now suppose $n > 2$. Prove that if Y_0 is any circle tangent to A and B, and Y_i are constructed inductively, for $i \geq 1$, so that Y_i is tangent to A, B and Y_{i-1}, then necessarily $Y_n = Y_0$, so the chain of circles closes up to form another constellation. Does the same result hold when $n = 2$?

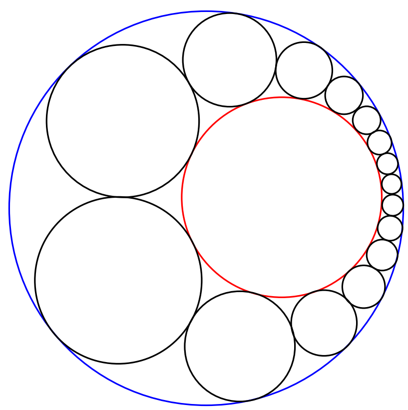

Figure 1: A constellation on the red and blue circles

4. Show from first principles that a vertical line segment is length-minimizing and hence defines a geodesic in the hyperbolic upper half-plane, i.e. the upper half-plane with the abstract metric $\frac{dx^2+dy^2}{y^2}$.

5. Let z_1, z_2 be distinct points in the upper half plane $\mathfrak{h}$. Suppose that the hyperbolic line through z_1 and z_2 meets the real axis at points w_1 and w_2, where z_1 lies on the hyperbolic line segment $w_1 z_2$ (and where one w_i may be ∞). Show that the hyperbolic distance $d_{hyp}(z_1, z_2) = \log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2 taken in an appropriate order.

6. (a) Let $P \in S^2 \subset \mathbb{R}^3$ be a point on the round sphere. The spherical circle with centre P and radius ρ is the set $\{w \in S^2 \,|\, d_{sph}(w, P) = \rho\}$, where d_{sph} is the spherical metric (induced by the first fundamental form of the embedding). Prove that a spherical circle of radius ρ is a Euclidean circle. Prove that its circumference is $2\pi \sin(\rho)$ and that it bounds a disc on S^2 of area $2\pi(1 - \cos(\rho))$.

   (b) Let $C \subset \mathfrak{h}$ be a hyperbolic circle with centre $p \in \mathfrak{h}$ and radius ρ, i.e. the locus $\{w \in \mathfrak{h} \,|\, d_{hyp}(w, p) = \rho\}$ for some $\rho > 0$. Show that C is a Euclidean circle. If $p = ic$ for $c \in \mathbb{R}_{>0}$, find the centre and radius of C as a Euclidean circle. Show the hyperbolic circumference of C is $2\pi \sinh(\rho)$, and the hyperbolic area of the disc it bounds is $2\pi(\cosh(\rho) - 1)$. Deduce that no hyperbolic triangle contains a hyperbolic circle of radius $> \cosh^{-1}(3/2)$.

   (c) Deduce that there is some $\delta > 0$ such that, in any hyperbolic triangle, the union of the δ-neighbourhoods of two of the sides completely contains the 3rd side. (Does such a δ exist for triangles in the Euclidean plane?)

7. Fix a hyperbolic triangle $\Delta \subset \mathbb{H}^2$ with interior angles A, B, C and side lengths (in the hyperbolic metric) a, b, c, where a is the side opposite the vertex with angle A, etc.

   (a) Suppose that C is a right-angle. By applying the 'hyperbolic cosine' formula in two different ways, prove that

$$\sin(A)\sinh(c) = \sinh(a).$$

   (b) Deduce that for a general hyperbolic triangle, one has

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$$

8. (a) Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel, and that in this case the common perpendicular is unique. Show that, up to isometry, for $t > 0$ there is a unique configuration of ultraparallel lines for which the segment of the common perpendicular between the lines has length t.

   (b) Let l_1, l_2 be ultraparallel hyperbolic lines, and let r_{l_i} denote the hyperbolic isometry given by reflection in l_i. Prove that $r_{l_1} \circ r_{l_2}$ has infinite order.

   (c)* Let l_1, l_2, l_3 be pairwise ultraparallel hyperbolic lines whose endpoints are cyclically ordered $l_1^+, l_1^-, l_2^+, l_2^-, l_3^+, l_3^-$ at infinity $\partial\mathbb{H}^2$. Let $T_a = r_{l_2} \circ r_{l_1}$ and $T_b = r_{l_3} \circ r_{l_2}$. Prove that T_a and T_b generate a *free* subgroup of the group of orientation-preserving isometries of the hyperbolic plane. [*Hint: if U is the region bound by the l_i, and $V = U \cup r_{l_2}U$, consider a 'tiling' of the plane by copies of V.*]

9. (a) Consider the 'ideal' hyperbolic square with vertices at $0, 1, \infty, -1$ in the upper half-plane model. By gluing the edges of the square by isometries, or otherwise, prove that there is a *complete* hyperbolic metric on the smooth surface $S^2 \backslash \{p, q, r\}$ given by the complement of 3 distinct points in the sphere.

   (b) Construct a non-orientable compact hyperbolic surface.

10. Let Σ be an abstract compact hyperbolic surface. Let γ_1 and γ_2 be simple *closed* geodesics on Σ, i.e. the images of smooth embeddings $\gamma_i : S^1 \to \Sigma$ which everywhere satisfy the geodesic equations. Prove that $\gamma_1 \sqcup \gamma_2$ cannot be the boundary of an embedded cylinder in Σ (i.e. a smooth subsurface homeomorphic to $S^1 \times [0, 1]$.

    Construct a compact abstract hyperbolic surface Σ, and disjoint simple closed geodesics $\gamma_i \subset \Sigma$, for which $\gamma_1 \sqcup \gamma_2$ bounds an embedded subsurface Σ' of Σ homeomorphic to the complement of two disjoint discs in a torus. Can this happen if Σ has genus two? Briefly justify your answer.

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