Geometry IB – 2020/21 – Sheet 3: *Geodesics and abstract Riemannian metrics* [circa Lectures 14–18]

1. Let $\Sigma = \{(x,y,z) \,|\, x^2 + y^2 = 1\}$ be the unit cylinder. Show that a geodesic on Σ through the point $(1,0,0)$ can be parametrized to be contained in a spiral of the form $\gamma(t) = (\cos\alpha t, \sin\alpha t, \beta t)$, where $\alpha^2 + \beta^2 = 1$.

2. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$. Suppose that a straight line $\ell = \mathbb{R} \subset \mathbb{R}^3$ lies entirely in Σ. Prove that ℓ is a geodesic on Σ. Deduce that through every point p of the hyperboloid $S = \{x^2 + y^2 = z^2 + 1\}$ there are (at least) three geodesics $\gamma_p : \mathbb{R} \to S$ defined on the entire real line $\mathbb{R}$.

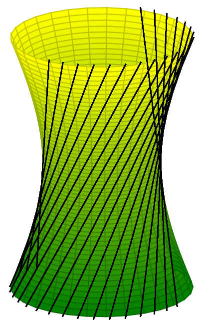

Figure 1: Lines on the hyperboloid of one sheet

3. Let $\Sigma \subset \mathbb{R}^3$ be a smooth surface of revolution in $\mathbb{R}^3$. Suppose the smooth curve $\gamma : (a,b) \to \Sigma$ satisfies the Clairaut condition
$$\rho(t) \cos\theta(t) = \text{constant}$$
where $\rho(t)$ is the distance from $\gamma(t)$ to the axis of revolution, and $\theta(t)$ is the angle between γ and the parallel at $\gamma(t)$. Suppose furthermore that there is no positive-length interval on which γ co-incides with a parallel. Show that γ is a geodesic. [*This gives a partial converse to the Clairaut relation.*]

4. (a) For $a > 0$, let Σ be the half-cone $\Sigma = \{(x,y,z) \,|\, z^2 = a(x^2 + y^2),\ z > 0\}$. Show that Σ is locally isometric to the Euclidean plane. By opening up the cone into a planar sector, or otherwise, show that when $a = 3$ no geodesic on Σ intersects itself, but for $a > 3$ there are geodesics which self-intersect.

   (b) Let γ be a geodesic on Σ which intersects the parallel $z = 1$ at an angle θ_0. Using Clairaut's relation, or otherwise, show that the smallest value of z obtained on γ is independent of a. What happens when $\theta_0 = \pi/2$?

5. Given an example of a connected smooth surface $\Sigma \subset \mathbb{R}^3$ and points $p, q \in \Sigma$ for which the infimum $\inf_\gamma L(\gamma)$ of lengths of piecewise smooth curves $\gamma : [a,b] \to \Sigma$ with $\gamma(a) = p$ and $\gamma(b) = q$ is strictly smaller than the length of any piecewise smooth curve γ between p and q.

6. Let $\eta : [s_0, s_1] \to \mathbb{R}^3$ be an embedded smooth curve parametrized by arc-length. Assume that $\eta''(s) \neq 0$ for every s (i.e. η has non-zero curvature). The *binormal vector* to η is the unit vector $b(s)$ in the direction $\eta'(s) \times \eta''(s)$. Consider the ruled surface with parametrization
$$\sigma(u,v) = \eta(u) + v b(u) \qquad u \in (s_0, s_1),\ -\varepsilon < v < \varepsilon,\ \text{where } \varepsilon > 0.$$
Prove that if ε is sufficiently small, then the image of σ defines a smooth surface in $\mathbb{R}^3$ on which η is a geodesic.

7. (a) The Möbius group $\mathbb{P}SL(2;\mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\}$. Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere, a smooth surface in $\mathbb{R}^3$. If we identify $\mathbb{C} \cup \{\infty\}$ with S^2 via stereographic projection, prove that the Möbius group acts on S^2 by diffeomorphisms.

   (b) Let $f : S^2 \to S^2$ be a diffeomorphism which is also a global isometry. By using that f sends geodesics to geodesics, or otherwise, show that f is the restriction to S^2 of an element of the orthogonal group $O(3)$.

   (c) If the Möbius map A defines an isometry of S^2, show that it commutes with the antipodal map $-1 : S^2 \to S^2$ (which sends $(x,y,z) \mapsto (-x,-y,-z)$). Is the converse true? Briefly justify your answer.

8. Show that the surfaces Σ and Σ' in $\mathbb{R}^3$ defined as the images of

$$\sigma(u,v) = (u\cos v, u\sin v, \ln u) \qquad \text{and} \qquad \tau(u,v) = (u\cos v, u\sin v, v)$$

(where $u > 0$ and $v > 0$) have the same Gauss curvature, but are not locally isometric.

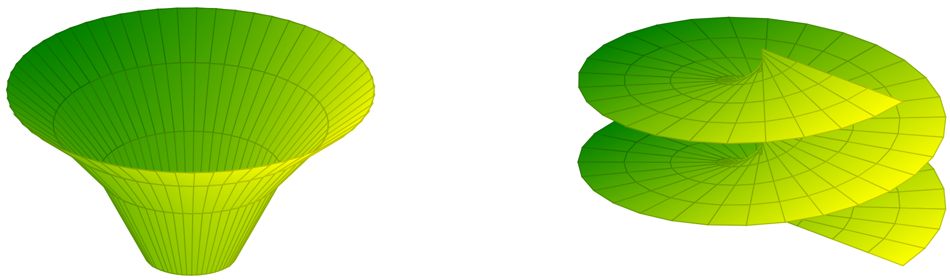

Figure 2: The surfaces Σ (left) and Σ' (right): an 'exponential cone' and a helicoid

9. (a) Define an abstract Riemannian metric on the disc $B(0,1) \subset \mathbb{R}^2$ by $\frac{du^2+dv^2}{1-u^2-v^2}$. Prove directly that diameters are then length-minimizing curves. Show that distances in the metric are bounded, but areas can be unbounded.

(b) Let $V \subset \mathbb{R}^2$ be the open square $V = \{|u| < 1,\ |v| < 1\}$. Define two abstract Riemannian metrics on V by

$$\frac{du^2}{(1-u^2)^2} + \frac{dv^2}{(1-v^2)^2} \qquad \text{and} \qquad \frac{du^2}{(1-v^2)^2} + \frac{dv^2}{(1-u^2)^2}.$$

Prove that the resulting surfaces are not isometric, but there is an area-preserving diffeomorphism between them. [*Hint: for the first statement, consider the lengths of curves going out to the boundary in the two surfaces.*]

10*. Consider a smooth surface $\Sigma \subset \mathbb{R}^3$ with Gauss curvature κ, and an allowable parametrization $\sigma : V \to U \subset \Sigma$ with first fundamental form $du^2 + G(u,v)dv^2$. Let $e = \sigma_u$, $f = \sigma_v/\sqrt{G}$ and $n = (\sigma_u \times \sigma_v)/\|\sigma_u \times \sigma_v\|$, so $\langle e, f, n\rangle$ form a 'moving frame', i.e. an orthonormal basis of $\mathbb{R}^3$ depending on the point $(u,v) \in V$.

- Show $n = e \times f$.
- Differentiating $e.e = 1 = f.f$ and $e.f = 0$, show there are constants $\alpha, \beta, \lambda_i, \mu_j$ for which

$$e_u = \alpha \cdot f + \lambda_1 \cdot n; \quad e_v = \beta \cdot f + \mu_1 \cdot n; \quad f_u = -\alpha \cdot e + \lambda_2 \cdot n; \quad f_v = -\beta \cdot e + \mu_2 \cdot n.$$

- Show $\alpha = 0$ and $\beta = \sqrt{G}_u$.
- Show $\lambda_1\mu_2 - \lambda_2\mu_1 = e_u \cdot f_v - f_u \cdot e_v = -\beta_u = -\sqrt{G}_{uu}$.
- Recalling that the Gauss map N satisfies $DN(\sigma_u) = n_u$ and $DN(\sigma_v) = n_v$ show $n_u \times n_v = \kappa \cdot (\sigma_u \times \sigma_v)$. By considering $(n_u \times n_v).n$, conclude that $\kappa \cdot \sqrt{G} = -\sqrt{G}_{uu}$.

Deduce Gauss' *theorema egregium*: if two smooth embedded surfaces in $\mathbb{R}^3$ are isometric, they have the same Gauss curvature.

11*. The first fundamental form makes sense for a surface $\Sigma \subset \mathbb{R}^n$ for any n. Let $S^1 \subset \mathbb{C}$ be the unit circle. Let $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ be the 'product torus'. Show that the induced metric on $S^1 \times S^1$ is locally Euclidean and hence flat. Show that through any point $p \in S^1 \times S^1$ there are infinitely many closed geodesics, and also infinitely many non-closed geodesics (defined on the whole of $\mathbb{R}$). [*A closed geodesic is a geodesic σ defined on the whole of $\mathbb{R}$ but which is periodic, so for some $L > 0$ we have $\sigma(t+L) = \sigma(t)$ for every $t \in \mathbb{R}$.*]

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