Geometry IB – 2020/21 – Sheet 2: *Smooth surfaces in* $\mathbb{R}^3$ [circa Lectures 8–13]

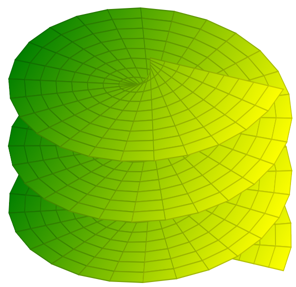
1. Show that a continuously differentiable curve $\gamma : (a, b) \to \mathbb{R}^3$ with $\gamma'(t) \neq 0$ for every t can be parametrized by arc-length.

2. *Mercator's projection* of the sphere is the chart whose inverse is the local parametrization

$$\sigma(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

Prove that this determines an allowable chart on the complement of a longitude, which sends lines of longitude and latitude to straight lines in the plane, and which preserves angles but not areas (*cf. Greenland versus Africa on a map; see also* `https://en.wikipedia.org/wiki/Mercator_projection`).

3. The *helicoid* is the ruled surface swept by a straight line which moves perpendicular to the z-axis and at time t passes through $(0, 0, t)$ and makes an angle t with the x-axis (consider the surface swept by the rotor blade of a helicopter rising vertically). Parametrize the helicoid and find its first fundamental form.



4. (a) Place the unit sphere $S^2 \subset \mathbb{R}^3$ inside a vertical circular cylinder C of radius one. Prove that horizontal projection from S^2 to C preserves area. Deduce that S^2 admits a smooth atlas of charts which are area-preserving.

   (b) A *lune* is one component of the region on the unit sphere S^2 cut out by two great circles (so it is a bigon). Prove that if the lune has internal angle α, it has area 2α. Hence, or otherwise, prove that a *spherical triangle*, i.e. a connected region bound by 3 great circles and with internal angles α, β, γ each less than π, has area $\alpha + \beta + \gamma - \pi$.

5. (a) Let $\eta : (a, b) \to \mathbb{R}^3$ be a smooth curve given by $\eta(u) = (f(u), 0, g(u))$. Suppose η' is never zero, η is a homeomorphism to its image and $f(u) > 0$ for all u. Let Σ denote the associated surface of revolution given by rotating η around the z-axis. Prove that the Gauss curvature κ of Σ is given by

$$\kappa = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}.$$

   If η is parametrized by arc-length, show $\kappa = -f''/f$.

   (b) Calculate κ for the hyperboloid of one sheet $\{x^2 + y^2 = z^2 + 1\}$ and of two sheets $\{x^2 + y^2 = z^2 - 1\}$. Describe the qualitative properties of κ (its sign, its behaviour near infinity). Illustrate the results with a picture.

6. Let $T \subset \mathbb{R}^3$ be the smooth embedded torus obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ in the xz-plane around the z axis. Sketch T, and draw an illustration of the places on T where (you believe) the Gauss curvature κ is positive, negative and zero.

   Now compute κ, check your illustration, and compute $\int_T \kappa \, dA$ (where dA is the area element associated to the metric on $T \subset \mathbb{R}^3$).

7. A compact smooth surface $\Sigma \subset \mathbb{R}^3$ is *strictly convex* if it is the boundary of a closed region $R \subset \mathbb{R}^3$ with the property that for any $x, y \in R$, the straight line segment $[x, y] \subset R$, and $[x, y] \cap \Sigma \subset \{x, y\}$. Show that the Gauss map of a strictly convex surface Σ is a bijection. If, furthermore, the Gauss curvature of Σ is everywhere positive, show that the Gauss map is a diffeomorphism, and deduce that $\int_\Sigma \kappa \, dA = 4\pi$, where κ is the Gauss curvature and dA is the area form on Σ. [*This is the 'convex' Gauss-Bonnet theorem.*]

8. Find the image of the Gauss map for the following smooth surfaces in $\mathbb{R}^3$:

   (a) $\Sigma_1 = \{x^2 + y^2 = z\}$;

   (b) $\Sigma_2 = \{x^2 + y^2 = \cosh\, z^2\}$.

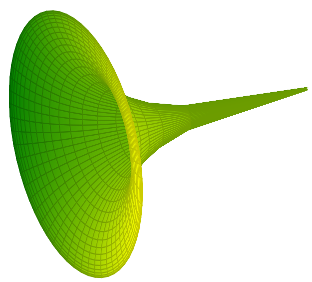
9. Let $\Sigma \subset \mathbb{R}^3$ be a smooth oriented surface, and $p \in \Sigma$. Let $n(p)$ be the unit normal vector at p. Let $v \in T_p\Sigma$ be a unit vector (with respect to the first fundamental form). Let γ_v be the plane curve which is the intersection of Σ and the affine two-plane $\mathbb{R}^2 = p + \mathrm{Span}\langle v, n(p)\rangle$. Viewing the second fundamental form as a bilinear form $I\!I_p$ on $T_p\Sigma$, show that $I\!I_p(v, v)$ is the curvature of the plane curve γ_v at p.

   [*The curvature of a plane curve* $\gamma : (a, b) \to \mathbb{R}^2$ *parametrized by arc-length is the function* $\kappa : (a, b) \to \mathbb{R}$ *for which* $\gamma''(s) = \kappa(s) n_\gamma(s)$, *with* $n_\gamma(s)$ *the unit normal to* γ *for which* $\langle \gamma'(s), n_\gamma(s)\rangle$ *forms a positively oriented basis for* $\mathbb{R}^2$.]

10*. The *tractrix* is the path followed by a heavy object which starts at $(1, 0)$ in $\mathbb{R}^2$ and is pulled by a person attached to the object by a (taut) rope of length 1 and who walks from the origin up the y-axis. The *tractoid* is the surface obtained by rotating the tractrix around the y-axis. Show that the tractrix can be described parametrically as

$$x = \sin t,\ \ y = (\cos t + \ln \tan(t/2)), \qquad t \in (\pi/2, \pi).$$

Prove that the tractoid is a smooth surface where $y > 0$, has Gauss curvature identically -1, and has total area 2π.



11*. (a) Let $\sigma : V \to \Sigma \subset \mathbb{R}^3$ be an allowable parametrization for a subset of a smooth surface in $\mathbb{R}^3$, with $V \subset \mathbb{R}^2$ a connected open subset with co-ordinate (u, v). Let $\alpha_{ijk} = \langle \sigma_i, \sigma_{jk}\rangle$ where $i, j, k \in \{u, v\}$ and subscripts denote partial differentiation. Show the functions α_{ijk} on V are determined by the first fundamental form.

(b) Let P be the matrix with columns σ_u, σ_v, n where n denotes the unit normal to Σ. Consider the matrix-valued functions on V defined by

$$C = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix};\ D_1 = \begin{pmatrix} \alpha_{uuu} & \alpha_{uvu} & -L \\ \alpha_{vuu} & \alpha_{vvu} & -M \\ L & M & 0 \end{pmatrix};\ D_2 = \begin{pmatrix} \alpha_{uuv} & \alpha_{uvv} & -M \\ \alpha_{vuv} & \alpha_{vvv} & -N \\ M & N & 0 \end{pmatrix}.$$

Show $P^t.P = C$, $P^t.P_u = D_1$ and $P^t.P_v = D_2$.

(c) Letting $A = C^{-1}D_1$ and $B = C^{-1}D_2$, deduce that each row ξ of P satisfies a system of linear first order partial differential equations

$$\xi_u = A^t\xi; \quad \xi_v = B^t\xi.$$

Assuming the *uniqueness* of solutions to such a differential system with a given initial condition, deduce that a connected smooth surface in $\mathbb{R}^3$ is determined up to rigid motion by its first and second fundamental forms.

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