Geometry IB – 2020/21 – Sheet 1: *Topological and smooth surfaces* [circa Lectures 1–7]

1. Show that a topological surface is connected if and only if it is path-connected.

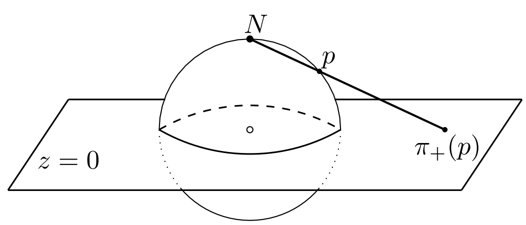
2. (a) Prove that the cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ is not a topological surface.

   (b) Let X be the space obtained from the disjoint union $\mathbb{R}^2_a \sqcup \mathbb{R}^2_b$ of two copies of the plane $\mathbb{R}^2$, labelled by a and b, by identifying $(x, y) \in \mathbb{R}^2_a$ with $(x, y) \in \mathbb{R}^2_b$ whenever $(x, y) \neq (0, 0)$. Show that X is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

   (c) Let Σ be a topological surface, and let $\mathbb{R}_\delta$ denote the real numbers with the *discrete* topology. Show that $\Sigma \times \mathbb{R}_\delta$ is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^2$, but which is not a topological surface.

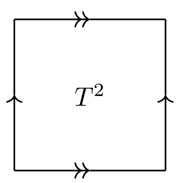
3. (a) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $\pi_\pm : S^2 \backslash \{(0, 0, \pm 1)\} \to \mathbb{R}^2 = \{z = 0\}$ denote stereographic projection from the north / south poles $(0, 0, \pm 1) \in S^2$. Show that the transition function between these two charts is given by $(u, v) \mapsto (u/(u^2 + v^2), v/(u^2 + v^2))$. Deduce that this atlas gives S^2 the structure of an abstract smooth surface.

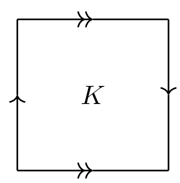
   (b) Now consider the chart on S^2 given by (U, ϕ) where $U = \{y < 0\}$ and $\phi : U \to \mathbb{R}^2$ maps $(x, y, z) \to (x, z)$. What is the image of ϕ? Check explicitly that this chart is *compatible* with the smooth atlas defined by $\pi_\pm$, i.e. the transition functions for the atlas with charts $\{\pi_+, \pi_-, \phi\}$ are all smooth.

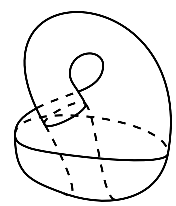


4. Define the torus T^2 as the quotient space of a square by the side identifications as shown on the left below. Prove that T^2 is homeomorphic to the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. [*Hint:* first show $S^1 \cong \mathbb{R}/\mathbb{Z}$.]

   Hence, or otherwise, show that the torus T^2 is homeomorphic to the surface of revolution in $\mathbb{R}^3$ defined by rotating the circle $\{(x - b)^2 + z^2 = a^2\}$ about the z-axis, where $0 < a < b$.







5. (a) The Klein bottle K is the quotient space of the square by the side identifications indicated above. By constructing a suitable atlas of charts, prove that K admits the structure of an abstract smooth surface.

   (b) Construct a continuous surjection $p : T^2 \to K$ such that for every $x \in K$, $p^{-1}(x)$ consists of exactly two points.

   (c) Draw inside the identification square for the Klein bottle K the set of points in which the 'usual' map $f : K \to \mathbb{R}^3$ (as drawn on the right above) fails to be injective. By considering a map $h = (f, \eta) : K \to \mathbb{R}^3 \times \mathbb{R}$, where η is a function of the width co-ordinate of the square, explain why K can be continuously embedded in $\mathbb{R}^4$ (i.e. there is a continuous map $K \to \mathbb{R}^4$ which is a homeomorphism to its image). [*You do not need an explicit formula for f. Many helpful pictures can be found at* `https://en.wikipedia.org/wiki/Klein_bottle`.]

6. Use cut-and-paste arguments to briefly explain why there is a homeomorphism between the topological surfaces $\mathbb{RP}^2 \# T^2$ and $\mathbb{RP}^2 \# K$, where $\#$ denotes the connect sum operation. [*See* `http://www.josleys.com/show_gallery.php?galid=377` *for an animation of the equivalence.*]

7. Let $H : \mathbb{R}^3 \to \mathbb{R}$ and $K : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Suppose the level sets $H^{-1}(0)$ and $K^{-1}(0)$ are smooth surfaces in $\mathbb{R}^3$ which intersect along a smooth curve γ in $\mathbb{R}^3$. Suppose $p = (x_0, y_0, z_0)$ belongs to γ, and that at p the expression $K_y H_z - K_z H_y$ is non-vanishing. Show that in some neighbourhood of p the curve γ can be locally parametrized by the variable x, and that if we write $\gamma(x) = (x, y(x), z(x))$ then

$$y'(x) = (K_z H_x - K_x H_z)/(K_y H_z - K_z H_y).$$

Verify this formula in the case $H = x^2 + y^2 - z$ and $K = x^2 + y^2 + z^2 - 1$.

8. Consider the subspace $\Sigma \subset \mathbb{R}^3$ defined by $\{(x, y, z) \in \mathbb{R}^3 \,:\, z = x^3 + y^3 - 3xy\}$. Observe that Σ is a smooth surface in $\mathbb{R}^3$ (*why?*). Show that the level set $\Sigma \cap \{z = c\}$ is a smooth curve in the plane unless $c \in \{-1, 0\}$. Sketch the level sets for values $c = -1$ and $c = 0$. [*For $c = -1$, it may help to factorise* $x^3+y^3-3xy+1 = (x+y+1)(x^2+y^2-xy-x-y+1)$.]

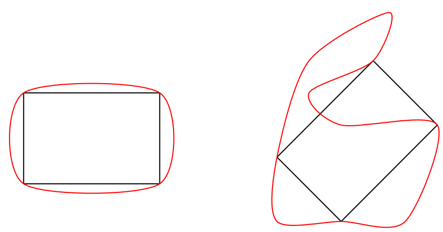
9. (a) Let X be the set of unordered pairs of points on a circle S^1. Explain why X is naturally a quotient space of the torus T^2. By considering T^2 as a quotient space of a square, as in question 4, or otherwise, show that X is homeomorphic to $\mathbb{RP}^2 \backslash \mathring{D}$, the complement of an open disc $\mathring{D} \subset \mathbb{RP}^2$ in $\mathbb{RP}^2$.

(b) Let $\phi : D = \{|z| \leq 1\} \to \mathbb{R}^2$ be a continuous injection of a closed disc D, with boundary $C = \phi(S^1) \subset \mathbb{R}^2$. Define a map $f : C \times C \to \mathbb{R}^3$ via

$$(u, v) \mapsto \left(\frac{u+v}{2}, |u - v|\right) \in \mathbb{R}^2 \times \mathbb{R}.$$

Show that f defines a map on X, which extends to a continuous map $\hat{\phi} : \mathbb{RP}^2 \to \mathbb{R}^3$.

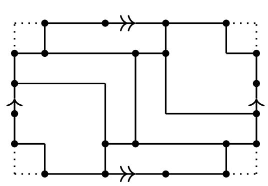
(c) Using without proof that a compact non-orientable topological surface cannot be continuously embedded in $\mathbb{R}^3$, deduce that C bounds an *inscribed rectangle*, i.e. there are four pairwise-distinct points on the curve which form the vertices of a rectangle in $\mathbb{R}^2$ (the rectangle need not be wholly contained in $\phi(D)$, cf. figure below). [*It is an open problem to show that every simple closed curve $C \subset \mathbb{R}^2$ bounds an inscribed* square.]



10. Consider a decomposition of a topological surface into polygons (with V vertices, E edges and F faces), where all vertices have valence ≥ 3 and every face contains ≥ 3 vertices. Let F_n denote the number of faces bound by precisely n edges, and V_m the number of vertices where precisely m edges meet. Show that $\sum_n n\, F_n = 2E = \sum_m m\, V_m$. If $V_3 = 0$, deduce $E \geq 2V$, whilst if $F_3 = 0$ deduce $E \geq 2F$. If the surface is a sphere, deduce that $V_3 + F_3 > 0$.

11*. (a) View a polygonal decomposition of a topological surface S with all vertices of valence ≥ 3 as a map drawn on S. The map can be coloured with N colours if we can colour faces so that faces which share an edge have different colours (but faces which meet only at a vertex may have the same colour). Suppose that $N \in \mathbb{N}$ is a positive integer such that $2E/F < N$ for every possible decomposition of S. Show that every map on S can be coloured with at most N colours. [*Hint: Induct on F, noting that the result is straightforward if $F < N$.*]

(b) Prove that for any decomposition of S (where all vertices have valence at least 3), we have $2E/F \leq 6\,(1 - \chi(S)/F)$, where $\chi(S)$ is the Euler characteristic of S. Deduce that any map on a torus can be coloured with 7 colours. The map on the torus depicted below can be coloured with 7 colours but not fewer – why? [*The black dots represent the vertices; note the 'corner' of the rectangle is not one.*]



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