Geometry IB – 2019/20 – Sheet 3: *Geodesics and abstract Riemannian metrics*

1. Let $\Sigma = \{(x, y, z) \,|\, x^2 + y^2 = 1\}$ be the unit cylinder. Show that a geodesic on Σ through the point $(1, 0, 0)$ can be parametrized to be contained in a spiral of the form $\gamma(t) = (\cos \alpha t, \sin \alpha t, \beta t)$, where $\alpha^2 + \beta^2 = 1$.

2. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$. Suppose that a straight line $\ell = \mathbb{R} \subset \mathbb{R}^3$ lies entirely in Σ. Prove that ℓ is a geodesic on Σ. Hence show that through every point p of the hyperboloid $S = \{x^2 + y^2 = z^2 + 1\}$ there is a geodesic $\gamma_p : \mathbb{R} \to S$ parametrized by arc-length and which is defined on the entire real line $\mathbb{R}$.

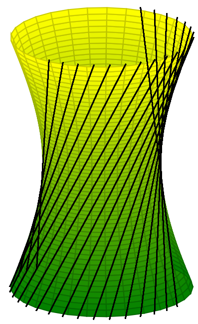

Figure 1: Lines on the hyperboloid of one sheet

3. Given an example of a connected smooth surface $\Sigma \subset \mathbb{R}^3$ and points $p, q \in \Sigma$ for which the infimum $\inf_\gamma L(\gamma)$ of lengths of piecewise smooth curves $\gamma : [a, b] \to \Sigma$ with $\gamma(a) = p$ and $\gamma(b) = q$ is strictly smaller than the length of any piecewise smooth curve γ between p and q.

4. (a) For $a > 0$, let Σ be the half-cone $\Sigma = \{(x, y, z) \,|\, z^2 = a(x^2 + y^2),\ z > 0\}$. Show that Σ is locally isometric to the Euclidean plane. By opening up the cone into a planar sector, or otherwise, show that when $a = 3$ no geodesic on Σ intersects itself, but for $a > 3$ there are geodesics which self-intersect.

   (b) Let γ be a geodesic on Σ which intersects the parallel $z = 1$ at an angle θ_0. Using Clairaut's relation, or otherwise, show that the smallest value of z obtained on γ is independent of a. What happens when $\theta_0 = \pi/2$?

5. A *surface of Liouville* is a smooth surface $\Sigma \subset \mathbb{R}^3$ which admits an allowable parametrization $\sigma(u, v)$ in which the first fundamental form has the shape

   $$E du^2 + 2F du dv + G dv^2 \quad \text{where} \quad E = G = U(u) + V(v),\ F = 0$$

   (so U respectively V are functions only of u respectively v). Show that the geodesics on such a surface can be written in the form

   $$\int \frac{du}{\sqrt{U - c}} = \pm \int \frac{dv}{\sqrt{V + c}} + c'$$

   for constants c, c' depending on the initial conditions (i.e. a point and tangent direction of the geodesic).

   If a geodesic γ makes an angle θ with a curve v =constant, show that $U \sin^2 \theta - V \cos^2 \theta$ is constant along γ.

6. Let $\eta : [s_0, s_1] \to \mathbb{R}^3$ be an embedded smooth curve parametrized by arc-length. Assume that η has non-zero curvature, so $\|\eta''(s)\| \neq 0$ for every s. The *binormal vector* to η is the unit vector $b(s)$ in the direction $\eta'(s) \times \eta''(s)$. Consider the ruled surface with parametrization

   $$\sigma(u, v) = \eta(u) + v b(u) \qquad u \in (s_0, s_1),\ -\varepsilon < v < \varepsilon,\ \text{where } \varepsilon > 0.$$

   Prove that if ε is sufficiently small, then the image of σ defines a smooth surface in $\mathbb{R}^3$ on which η is a geodesic.

7. Define an abstract Riemannian metric on the disc $B(0, 1) \subset \mathbb{R}^2$ by $\frac{du^2 + dv^2}{1 - u^2 - v^2}$. Prove directly that diameters are then length-minimizing curves. Show that distances in the metric are bounded, but areas can be unbounded.

8. (a) Let $V \subset \mathbb{R}^2$ be the open square $V = \{|u| < 1, |v| < 1\}$. Define two abstract Riemannian metrics on V by

$$\frac{du^2}{(1-u^2)^2} + \frac{dv^2}{(1-v^2)^2} \qquad \text{and} \qquad \frac{du^2}{(1-v^2)^2} + \frac{dv^2}{(1-u^2)^2}.$$

Prove that the resulting surfaces are not isometric, but there is an area-preserving diffeomorphism between them. [*Hint: for the first statement, consider the lengths of curves going out to the boundary in the two surfaces.*]

(b) Show that the surfaces Σ and Σ' in $\mathbb{R}^3$ defined as the images of

$$\sigma(u,v) = (u\cos v, u\sin v, \ln u) \qquad \text{and} \qquad \tau(u,v) = (u\cos v, u\sin v, v)$$

(where $u > 0$ and $v > 0$) have the same Gauss curvature, but are not locally isometric.

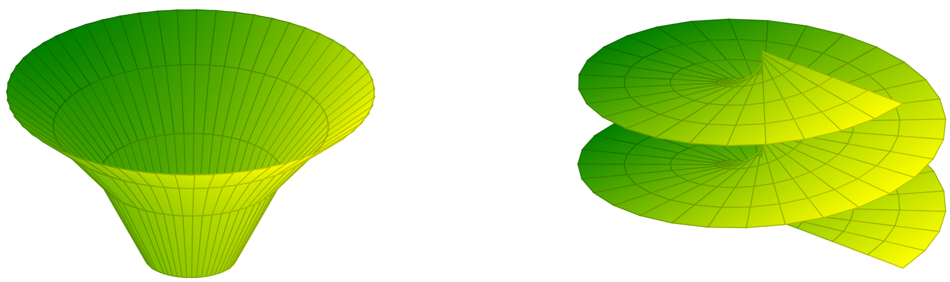

Figure 2: The surfaces Σ (left) and Σ' (right): an 'exponential cone' and a helicoid

9. (a) The Möbius group $\mathbb{P}SL(2;\mathbb{C})$ acts on $S^2 = \mathbb{C} \cup \{\infty\}$. Prove that this is an action by smooth diffeomorphisms, if we identify $\mathbb{C} \cup \{\infty\}$ with the round unit sphere $S^2 \subset \mathbb{R}^3$ via stereographic projection.

(b) Let $f : S^2 \to S^2$ be a global isometry of the unit sphere. By using that f sends geodesics to geodesics, or otherwise, show that f is the restriction to S^2 of an element of the orthogonal group $O(3)$.

(c) If the Möbius map A defines an isometry of S^2, show that it commutes with the antipodal map $-1 : S^2 \to S^2$ (which sends $(x,y,z) \mapsto (-x,-y,-z)$). Is the converse true? Briefly justify your answer.

10. The first fundamental form makes sense for a surface $\Sigma \subset \mathbb{R}^n$ for any n. Let $S^1 \subset \mathbb{C}$ be the unit circle. Let $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ be the 'product torus'. Show that the induced metric on $S^1 \times S^1$ is locally Euclidean and hence flat. Show that through any point $p \in S^1 \times S^1$ there are infinitely many closed geodesics, and also infinitely many non-closed geodesics.

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