Geometry IB – 2019/20 – Sheet 2: *Smooth surfaces in* $\mathbb{R}^3$

1. Show that a differentiable curve $\gamma : (a,b) \to \mathbb{R}^3$ with $\gamma'(t) \neq 0$ for every t can be parametrized by arc-length.

2. *Mercator's projection* of the sphere is the chart whose inverse is the local parametrization

$$\sigma(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

Prove that this determines a chart on the complement of a longitude, which sends lines of longitude and latitude to straight lines in the plane, and which preserves angles but not areas (*cf. Greenland versus Africa on a map; see also* `https://en.wikipedia.org/wiki/Mercator_projection`).

3. The *helicoid* is the ruled surface swept by a straight line which moves perpendicular to the z-axis and at time t passes through $(0,0,t)$ and makes an angle t with the x-axis (consider the surface swept by the rotor blade of a helicopter rising vertically). Parametrize the helicoid and find its first fundamental form.

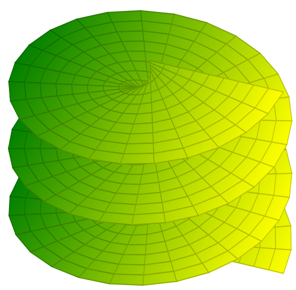

Figure 1: The helicoid

4. (a) Let $\eta : (a,b) \to \mathbb{R}^3$ be a smooth curve given by $\eta(u) = (f(u), 0, g(u))$. Suppose η' is never zero, η is a homeomorphism to its image and $f(u) > 0$ for all u. Let Σ denote the associated surface of revolution given by rotating η around the z-axis. Prove that the Gauss curvature κ of Σ is given by

$$\kappa = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}.$$

If η is parametrized by arc-length, show $\kappa = -f''/f$. Verify from this viewpoint that the unit sphere has constant Gauss curvature $+1$. What about the round sphere $S^2 \subset \mathbb{R}^3$ of radius R?

(b) Calculate the Gauss curvature κ of the hyperboloid of one sheet $\{x^2 + y^2 = z^2 + 1\}$ and of two sheets $\{x^2 + y^2 = z^2 - 1\}$. In both cases, describe the qualitative properties of κ (its sign, its behaviour near infinity). Illustrate the results with a picture.

5. Place the unit sphere $S^2 \subset \mathbb{R}^3$ inside a vertical circular cylinder C of radius one. Prove that horizontal projection from S^2 to C preserves area. Deduce that S^2 admits a smooth atlas of charts which are area-preserving.

6. A *lune* is the region on the unit sphere S^2 cut out by two great circles. Prove that if these circles meet at angle α, the lune has area 2α. Hence, or otherwise, prove that a *spherical triangle*, i.e. a connected region bound by 3 great circles and with internal angles α, β, γ each less than π, has area $\alpha + \beta + \gamma - \pi$.

7. Let $T \subset \mathbb{R}^3$ be the smooth embedded torus obtained by rotating the circle $(x-2)^2 + z^2 = 1$ in the xz-plane around the z axis. Calculate the surface area of T.

Compute the Gauss curvature κ of T. Draw an illustration of the places on T where κ is positive, negative and zero respectively, and verify that $\int_T \kappa \, dA = 0$ (where dA is the area element associated to the metric on $T \subset \mathbb{R}^3$).

8. The *tractrix* is the path followed by a heavy object which starts at $(1, 0)$ in $\mathbb{R}^2$ and is pulled by a person attached to the object by a (taut) rope of length 1 and who walks from the origin up the y-axis. The *tractoid* is the surface obtained by rotating the tractrix around the y-axis. Show that the tractrix can be described parametrically as

$$x = \sin t, \;\; y = (\cos t + \ln \tan(t/2)), \qquad t \in (\pi/2, \pi).$$

Prove that the tractoid is a smooth surface where $y > 0$, has Gauss curvature identically -1, and has total area 2π.

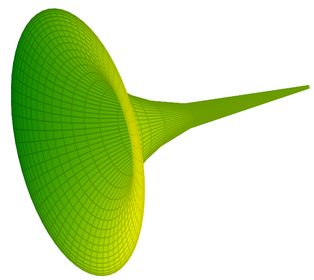

Figure 2: The tractoid

9. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded oriented surface, and $p \in \Sigma$. Let $n(p)$ be the unit normal vector at p. Let $v \in T_p\Sigma$ be a unit vector (with respect to the first fundamental form). Let γ_v be the plane curve which is the intersection of Σ and the two-plane $\mathbb{R}^2 = \langle n(p), v\rangle$. Viewing the second fundamental form as a bilinear form $I\!I_p$ on $T_p\Sigma$, show that $I\!I_p(v, v)$ is the curvature of the plane curve γ_v at p.

[*The curvature of a plane curve* $\gamma : (a, b) \to \mathbb{R}^2$ *parametrized by arc-length is the function* $\kappa : (a, b) \to \mathbb{R}$ *for which* $\gamma''(s) = \kappa(s) n_\gamma(s)$*, with* $n_\gamma(s)$ *the unit normal to* γ *for which* $\langle \gamma'(s), n_\gamma(s)\rangle$ *forms a positively oriented basis for* $\mathbb{R}^2$.]

Ivan Smith
is200@cam.ac.uk