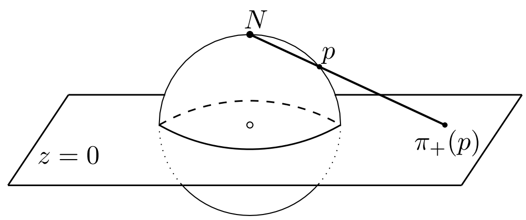
Geometry IB – 2019/20 – Sheet 1: *Topological and smooth surfaces*

1. (a) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Let $\pi_+ : S^2\backslash\{N\} \to \mathbb{R}^2 = \{z = 0\}$ denote stereographic projection from the north pole $N = (0, 0, 1) \in S^2$, and π_- stereographic projection from the south pole $(0, 0, -1)$. Show that the transition function between these two charts is given by $(u, v) \mapsto (u/(u^2 + v^2), v/(u^2 + v^2))$. Deduce that S^2 is a smooth surface.

   (b) Now consider the chart on S^2 given by (U, ϕ) where $U = \{y < 0\}$ and $\phi : U \to \mathbb{R}^2$ maps $(x, y, z) \to (x, z)$. What is the image of ϕ? Check explicitly that this chart is *compatible* with the smooth atlas defined by $\pi_\pm$, i.e. the transition functions for the atlas with charts $\{\pi_+, \pi_-, \phi\}$ are all smooth.



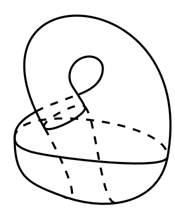
2. Show that an open subset of $\mathbb{R}^2$ is path-connected if and only if it is connected. Deduce that a topological surface is connected if and only if it is path-connected.

3. (a) Prove that the cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$ is not a topological surface.

   (b) Let X be the space obtained from the disjoint union $\mathbb{R}^2_a \sqcup \mathbb{R}^2_b$ of two copies of the plane $\mathbb{R}^2$, labelled by a and b, by identifying $(x, y) \in \mathbb{R}^2_a$ with $(x, y) \in \mathbb{R}^2_b$ whenever $(x, y) \neq (0, 0)$. Show that X is locally homeomorphic to $\mathbb{R}^2$, but is not a topological surface.

4. Define the torus T^2 as the quotient space of a square by the side identifications as shown on the left below. Prove that T^2 is homeomorphic to the quotient space $\mathbb{R}^2/\mathbb{Z}^2$.

   Hence, or otherwise, show that the torus T^2 is homeomorphic to the surface of revolution defined by rotating the circle $\{(x - b)^2 + z^2 = a^2\}$ about the z-axis, where $0 < a < b$.



5. (a) The Klein bottle K is the quotient space of the square by the side identifications indicated above. By constructing a suitable atlas of charts, prove carefully that K is a smooth surface.

   (b) Construct a continuous surjection $p : T^2 \to K$ such that for every $x \in K$, $p^{-1}(x)$ consists of exactly two points.

   (c) Draw inside the identification square for the Klein bottle K the set of points in which the 'usual' map $f : K \to \mathbb{R}^3$ (as drawn on the right above) fails to be injective. By considering a map $h = (f, \eta) : K \to \mathbb{R}^3 \times \mathbb{R}$, where η is a function of the width co-ordinate of the square, explain why K can be continuously embedded in $\mathbb{R}^4$ (i.e. there is a continuous map $K \to \mathbb{R}^4$ which is a homeomorphism to its image). [*You do not need an explicit formula for f. Many helpful pictures can be found at* `https://en.wikipedia.org/wiki/Klein_bottle`.]

6. State the implicit function theorem. For $k \in \{1, 2\}$, if $F : \mathbb{R}^3 \to \mathbb{R}^k$ satisfies that $DF|_p$ is onto for some $p \in \mathbb{R}^3$, show that there are (smooth, invertible) local co-ordinate changes defined near p and $F(p) = q$ after which F locally agrees with one of the projection maps $(x, y, z) \mapsto x$ or $(x, y, z) \mapsto (x, y)$.

   Deduce that, for fixed q, the set of solutions $Z_q = \{u \in \mathbb{R}^3 \,|\, F(u) = q\}$ is locally a smooth curve ($k = 2$) or surface ($k = 1$) near p.

7. Consider the subspace $\Sigma \subset \mathbb{R}^3$ defined by $\{(x,y,z) \in \mathbb{R}^3 : z = x^3 + y^3 - 3xy\}$.

   (a) Prove that Σ is a smooth surface.

   (b) Show that the level set $\Sigma \cap \{z = c\}$ is a smooth curve in the plane unless $c \in \{-1, 0\}$. Sketch the level sets for values $c = -1$ and $c = 0$. [*For $c = -1$, it may help to factorise $x^3+y^3-3xy+1 = (x+y+1)(x^2+y^2-xy-x-y+1)$.*]

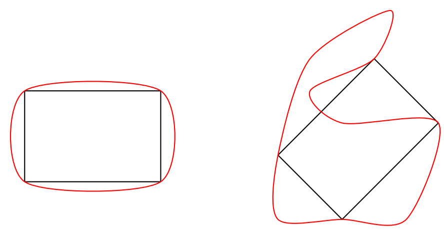
8. (a) Let X be the set of unordered pairs of points on a circle S^1. Explain why X is naturally a quotient space of the torus T^2. By considering T^2 as a quotient space of a square, as in question 4, or otherwise, show that X is homeomorphic to $\mathbb{RP}^2 \backslash \mathring{D}$, the complement of an open disc $\mathring{D} \subset \mathbb{RP}^2$ in $\mathbb{RP}^2$.

   (b) Let $\phi : D = \{|z| \leq 1\} \to \mathbb{R}^2$ be a continuous injection of a closed disc D, with boundary $C = \phi(S^1) \subset \mathbb{R}^2$. Define a map $f : C \times C \to \mathbb{R}^3$ via

$$(u,v) \mapsto \left(\frac{u+v}{2}, |u-v|\right) \in \mathbb{R}^2 \times \mathbb{R}.$$

   Show that f defines a map on X, which extends to a continuous map $\hat{\phi} : \mathbb{RP}^2 \to \mathbb{R}^3$.

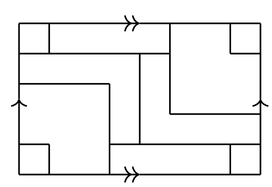
   (c) Using without proof that a compact non-orientable topological surface cannot be continuously embedded in $\mathbb{R}^3$, deduce that C bounds an *inscribed rectangle*, i.e. there are four pairwise-distinct points on the curve which form the vertices of a rectangle in $\mathbb{R}^2$ (the rectangle need not be wholly contained in $\phi(D)$, cf. figure below). [*It is an open problem to show that every simple closed curve $C \subset \mathbb{R}^2$ bounds an inscribed* square.]



9. Consider a decomposition of a topological surface into polygons (with V vertices, E edges and F faces), where all vertices have valence ≥ 3. Let F_n denote the number of faces bound by precisely n edges, and V_m the number of vertices where precisely m edges meet. Show that $\sum_n n\,F_n = 2E = \sum_m m\,V_m$. If $V_3 = 0$, deduce $E \geq 2V$, whilst if $F_3 = 0$ deduce $E \geq 2F$. If the surface is a sphere, deduce that $V_3 + F_3 > 0$.

10. (a) Viewing a polygonal decomposition of a topological surface S as a map drawn on S, we say that the map can be coloured with N colours if we can colour faces so that faces which share an edge have different colours (but faces which meet only at a vertex may have the same colour). Suppose that $N \in \mathbb{N}$ is a positive integer such that $2E/F < N$ for every possible decomposition of S. Show that every map on S can be coloured with at most N colours. [*Hint: Induct on F, noting that the result is straightforward if $F < N$.*]

    (b) Prove that for any decomposition of S (where all vertices have valence at least 3), we have $2E/F \leq 6\,(1 - \chi(S)/F)$, where $\chi(S)$ is the Euler characteristic of S. Deduce that any map on a torus can be coloured with 7 colours. The map on the torus depicted below can be coloured with 7 colours but not fewer – why?



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