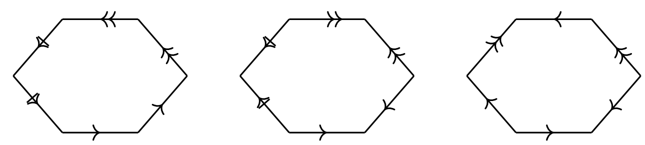
Geometry IB – 2019/20 – *Revision sheet*

These questions are not intended to be representative of the whole course, but to give a few practice questions for parts of the course not explicitly in the previous syllabus (e.g. topological surfaces; orientability; implicit function theorem; Gauss map; hyperbolic surfaces). Many past paper Geometry IB questions are also appropriate for this course. [*These questions are to help gain familiarity with the new topics: this is not a mock exam, and the questions are not moderated to the usual exam question style.*]

1. Define a *topological surface*. Give an example of a topological space which is locally homeomorphic to the plane, but which is not a topological surface.

   The gluing patterns below represent quotient spaces of a planar hexagon, where edges marked with the same label are glued together by an isometry of the corresponding intervals (respecting the indicated orientation). For each of the three cases, explain whether the resulting topological space is a topological surface, and if so, whether or not it is orientable. Justify your answers. [*You may assume that if D is homeomorphic to an open disc and $\gamma : (0,1) \to D$ is a continuous embedding, the complement $D \backslash \mathrm{im}(\gamma)$ has at most two connected components.*]

2. Define the *Klein bottle* K as a topological surface. Show that there are circles C_1 and C_2 in K such that $K \backslash C_1$ is connected but $K \backslash C_2$ is not connected. In the latter case, are the connected components of $K \backslash C_2$ necessarily pairwise homeomorphic? Can they be pairwise homeomorphic? Justify your answers.

   By gluing the sides of an ideal hyperbolic polygon by isometries, or otherwise, prove that the complement $K \backslash \{p\}$ of a point in the Klein bottle admits an abstract Riemannian metric of constant curvature -1. What is its area? Can K itself admit a metric of constant curvature -1? Briefly justify your answers.

3. Define an *abstract smooth surface*.

   Construct a Möbius band B by taking an open rectangular strip of paper $R = (-1,1) \times [-2,2]$ and gluing the ends with a half-twist:
   $$(2,t) \sim (-2,-t) \qquad \forall t \in (-1,1).$$
   The 'equator' of the Möbius band is the circle $C \subset B$ which is the image of the line $\{0\} \times [-2,2] \subset R$. Prove that B admits an abstract Riemannian metric g_B which is flat, and for which the equator C is a geodesic.

   Let $\iota : B \hookrightarrow \mathbb{R}^3$ be a smooth embedding for which $\iota(C)$ is a circle in the xy-plane. Can the first fundamental form of B with respect to this embedding agree with g_B? Justify your answer.

4. The *torus* T^2 is the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. Prove that T^2 admits the structure of a topological surface, and that T^2 is homeomorphic to the product space $S^1 \times S^1$ where $S^1 = \{|z| = 1\} \subset \mathbb{C}$.

   Define the *disc model* (D, g_{hyp}) of the hyperbolic plane, and state a theorem describing its isometries. Show that each such isometry defines a homeomorphism of the circle at infinity in hyperbolic space. By considering an action on (the end-points at infinity of) oriented geodesics, or otherwise, deduce that there is a non-trivial homomorphism of groups $\mathrm{Isom}(D, g_{hyp}) \to \mathrm{Homeo}(T^2)$ from the isometry group of the hyperbolic plane to the group of self-homeomorphisms of the torus.

   Let R_θ denote the homeomorphism of S^1 given by rotation by $\theta \in [0, 2\pi)$. For which pairs (θ, ϕ) does the homeomorphism $R_\theta \times R_\phi$ of $T^2 = S^1 \times S^1$ arise from a hyperbolic isometry? Justify your answer.

5. (a) State the *inverse function theorem*. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and $\Sigma = f^{-1}(0) \subset \mathbb{R}^3$. Prove that if $Df_x \neq 0$ for every $x \in \Sigma$, then Σ is locally given by the graph of a smooth function over one of the co-ordinate planes in $\mathbb{R}^3$.

   (b) Let f be a smooth function of one variable, and define
   $$\sigma(u,v) = (u, v, uf(v/u)) \qquad u \neq 0.$$
   Prove that σ defines a smooth surface $\Sigma \subset \mathbb{R}^3$, and that at every point $p \in \Sigma$ the (affine) tangent plane $T_p\Sigma$ passes through the origin.

6. State the *implicit function theorem.*

   Which of the following subsets $S_i \subset \mathbb{R}^3$ is a smooth embedded surface in $\mathbb{R}^3$? Justify your answers.

   (a) $S_1 = \{x^3 + y^3 + z^3 = 1\}$.

   (b) $S_2 = \{xyz = 0\}$.

   (c) $S_3 = \{x^2 + y^2 = \cosh(z^2)\}$.

   Give an example of a smooth surface $S \subset \mathbb{R}^3$ which is not of the form $f^{-1}(0)$ for any smooth function $f : \mathbb{R}^3 \to \mathbb{R}$.

7. Let $\eta : \mathbb{R} \to \mathbb{R}^2 = \{y = 0\} \subset \mathbb{R}^3$ be a smooth embedded curve parametrized by arc length and with $\eta(t) = (f(t), 0, g(t))$ with $f > 0$. Define the *surface of revolution* Σ_η associated to η. Define the *Gauss map* of an oriented smooth embedded surface in $\mathbb{R}^3$.

   Let $n : \Sigma_\eta \to S^2$ be the Gauss map. Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation by angle θ about the z-axis. Prove that for $p \in \Sigma_\eta$, $n(R_\theta(p)) = R_\theta(n(p))$.

   Hence, or otherwise, prove that for the hyperboloid $\{x^2 + y^2 = z^2 + 1\}$, the image of the Gauss map is the open annulus $\{|z| < 1/\sqrt{2}\} \subset S^2$.

8. Let $\Sigma \subset \mathbb{R}^3$ be a smooth embedded surface in $\mathbb{R}^3$. Prove that near any point $p \in \Sigma$, the surface Σ is given locally as the graph of a smooth function over the (affine) tangent plane $T_p\Sigma$. (You may use any version of the implicit function theorem, but should carefully state it.)

   Let $\Sigma \subset \mathbb{R}^3$ be a compact smooth embedded surface in $\mathbb{R}^3$. Let $P = \{x \in \Sigma \,|\, \kappa(x) \geq 0\}$ be the subset of points of Σ where the Gauss curvature is non-negative. By considering the family of planes in $\mathbb{R}^3$ whose normal vector is a given point of the unit sphere S^2, or otherwise, prove that the restriction of the Gauss map to P is surjective.

9. Define a *ruled surface* in $\mathbb{R}^3$, and what it means for a ruled surface to be *developable*. Give examples of ruled surfaces which are, respectively are not, developable, briefly justifying your answers.

   Consider the surface $\Sigma \subset \mathbb{R}^3$ with parametrization

   $$\sigma(u, v) = \gamma(u) + v\, a(u) \qquad u \in [0, 2\pi), v \in (-1, 1)$$

   where $\gamma(u) = (\cos u, \sin u, 0)$ and $a(u) = (\cos(u/2)\cos(u), \cos(u/2)\sin(u), \sin(u/2))$. Sketch Σ, and prove that Σ is a smooth surface for which the Gauss curvature κ is everywhere negative.

   Is there a smooth embedded compact surface S in $\mathbb{R}^3$ which contains a non-empty open subset $U \subset S$ with $\kappa < 0$ on U? Is there such a surface S which contains Σ as an open subset? Briefly justify your answers.

Ivan Smith
is200@cam.ac.uk